Dimensional Reduction for Paraxial Beams with Thermally Mediated Nonlocal Nonlinearity

Francesco Lorenzi[1,3], Luca Salasnich[2,3]

[1]Department of Information Engineering, University of Padova
[2]Department of Physics and Astronomy "Galileo Galilei", University of Padova
[3]INFN, Sezione di Padova

XXVI Riunione Nazionale di Elettromagnetismo – L'Aquila, 6–9 September 2026

Session T1B, Optics and Photonics, Tuesday 8 September, 09:45

UNIVERSITÀ DI PADOVA
Dipartimento
di Ingegneria
dell'Informazione

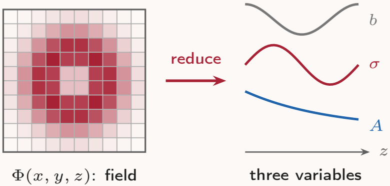
Dipartimento di Fisica e
Astronomia
"Galileo Galilei"

INFN

Dimensional reduction for quasi one-dimensional nonlinear problems

1. **Select an ansatz.** A trial field with a few parameters, e.g. a width and a phase curvature.
2. **Integrate the action** over the transverse plane, obtaining a Lagrangian in those parameters.
3. **Apply the variational principle.** The Euler–Lagrange equations in the parameters are a few ordinary differential equations.

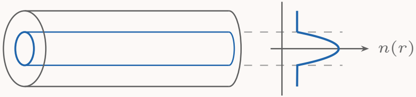
- **Advantage**: a model with few variables and an **effective potential**. **Limitation**: the transverse profile is prescribed by the ansatz.

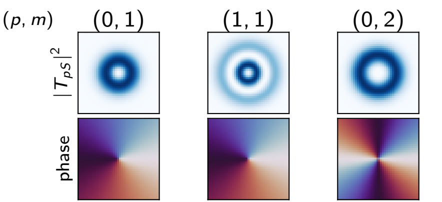


Established applications: pulses in optical fibres,
[D. Anderson, Phys. Rev. A **27**, 3135–3145 (1983)]
[B. A. Malomed, *Progress in Optics* (2002)]
trapped condensates and the nonpolynomial Schrödinger equation,
[V. M. Pérez-García et al., Phys. Rev. Lett. **77**, 5320–5323 (1996)]
[L. Salasnich et al., Phys. Rev. A **65**, 043614 (2002)]
beams in graded-index media.
[**FL** et al., Nanophotonics **14**, 805–813 (2025)]

$$\mathrm{i}\,\partial_z\Phi = \Big[-\tfrac{1}{2}\nabla_\perp^2 \underbrace{-\tfrac{1}{2}\,\partial_t^2}_{\text{anomalous dispersion}} + \underbrace{W(x,y)}_{\text{graded index}} + \underbrace{g\,|\Phi|^2}_{\text{Kerr}}\Big]\Phi, \qquad \Phi = \Phi(x,y,z,t).$$

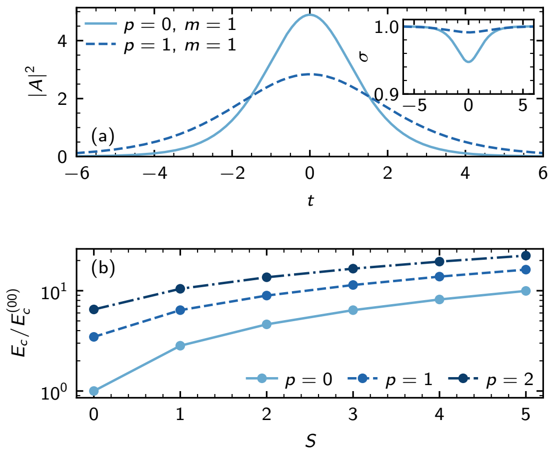
- A short pulse in a **graded-index fibre**: a $(3+1)$-D nonlinear Schrödinger equation.
  [FL et al., Nanophotonics 14, 805–813 (2025)]
- Φ is the envelope; lengths in the waist w_0, z in $L_D = kw_0^2$, t in $T_0 = \sqrt{|\beta_2| L_D}$: dispersion and diffraction share L_D.
- For $g < 0$, formally the **Gross–Pitaevskii equation** of an attractive condensate, with z in the role of time: the **nonpolynomial Schrödinger approach** applies.
  [L. Salasnich et al., Phys. Rev. A 65, 043614 (2002)]
- **Ansatz**: one Laguerre–Gauss mode, $\Phi = A\,T_{pS}(r;\sigma)\,\mathrm{e}^{\mathrm{i}m\theta}/\sqrt{2\pi}$, so $|A|^2$ is the power; A and σ depend on z and t.





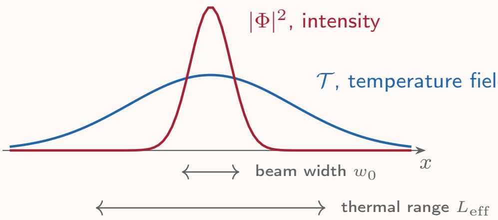
$$\mathrm{i}\,\partial_z A = -\tfrac{1}{2}\,\partial_t^2 A + \xi_{pS}\,\frac{1+\frac{3}{2}\,g_{pS}|A|^2}{\sqrt{1+g_{pS}|A|^2}}\,A$$

- Substitute the width back: **one** $(1+1)$**-D equation** for the pulse.
- **(a)** Solitons in **closed form**; the width dips at the peak of the pulse (inset).
- **(b)** Solitons exist up to a **critical energy** E_c, larger for higher modes.
- **Vakhitov–Kolokolov**: for solitons $A = a(t)\,\mathrm{e}^{-\mathrm{i}\kappa z}$, stable where $dE/d\kappa < 0$.
- **Experiment**: multimode solitons observed in a graded-index fibre, femtosecond pulses at telecom wavelength over many dispersion lengths.
  [W. H. Renninger et al., Nat. Commun. 4, 1719 (2013)]



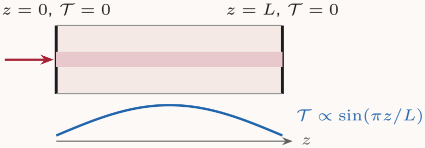
2) Beams in a thermal medium: the index change spreads

- **Thermal effects.** Absorption heats the medium; the heat diffuses, following the **Fourier heat equation**.
- $\Delta n = \beta\mathcal{T}$, $\beta = dn/d\mathcal{T}$ the **thermorefractive index**, large in lead glass, silicon, liquid crystals: the index follows $\mathcal{T}$ over a **surrounding region**.
- A nonlocal response **arrests collapse**; a long enough range **stabilises ring beams**.
  [O. Bang et al., Phys. Rev. E **66**, 046619 (2002)]
  [A. I. Yakimenko et al., Phys. Rev. E **71**, 065603 (2005)]
- **Realised experimentally** with beams in lead glass and in cells of nematic liquid crystals: self-trapped beams and vortex solitons.
  [C. Rotschild et al., Phys. Rev. Lett. **95**, 213904 (2005)]
  [C. Conti et al., Phys. Rev. Lett. **92**, 113902 (2004)]

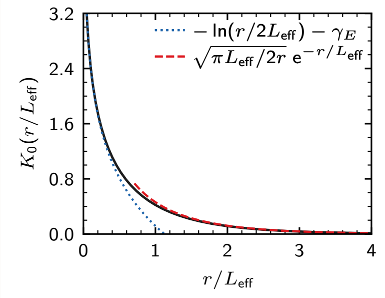


From heat diffusion to a Bessel kernel

$$\left(-\nabla_\perp^2 + L_{\text{eff}}^{-2}\right)\mathcal{T} = \eta\,|\Phi|^2 \quad\Longrightarrow\quad \mathcal{T} = \eta\, G * |\Phi|^2, \qquad G(r) = \frac{1}{2\pi}\,K_0\!\left(\frac{r}{L_{\text{eff}}}\right), \qquad L_{\text{eff}} = \frac{L}{\pi}.$$



- **Steady heat equation**, absorbed power $\alpha_0|\Phi|^2$ as source: $\eta = \alpha_0/(\rho\, c_p\, \kappa)$ (density, specific heat, diffusivity).
- Facets at **fixed temperature**: the longitudinal mode $\sin(\pi z/L)$ adds the **screening** term L_{eff}^{-2}.
- The kernel K_0 is **logarithmic** below L_{eff} and decays **exponentially** above it; reorientational and electrostrictive nonlocalities obey the same screened Poisson equation, each with its own L_{eff}.
  [A. Alberucci et al., Phys. Rev. A **91**, 013841 (2015)]
  [C. Conti et al., Phys. Rev. Lett. **95**, 183902 (2005)]

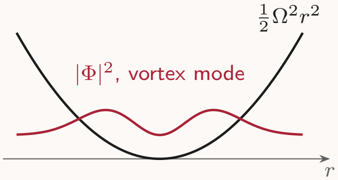


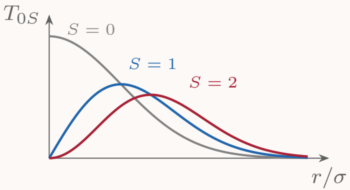
$$\mathrm{i}\,\partial_z \Phi = \left(-\mathrm{i}\frac{\alpha_0}{2} - \frac{1}{2}\nabla_\perp^2 + \frac{\Omega^2 r^2}{2} + \gamma\,\Theta[\,|\Phi|^2\,]\right)\Phi, \qquad \Theta[n] = G * n.$$

- Φ is the envelope; lengths in the waist w_0, z in the diffraction length $L_D = k w_0^2$, k the wavenumber; power $P = \int |\Phi|^2 d^2\mathbf{r}$.
- $\gamma \propto -\beta\,\eta$ ties the nonlinearity to the absorption: $\gamma > 0$ **defocusing** ($\beta < 0$), $\gamma < 0$ **focusing**.
  [FL et al., Front. Phys. 21, 102203 (2026)]
- **Ansatz**: one annular mode with a radial chirp,

$$\Phi(r,\theta,z) = A(z)\,T_{pS}\big(r;\sigma(z)\big)\;\exp\!\Big[\mathrm{i}\,\frac{b(z)}{2}\,r^2\Big]\;\mathrm{e}^{\mathrm{i}m\theta}.$$

- $\xi_{pS} = S + 2p + 1$, with $S = |m|$ the charge and p the radial index.





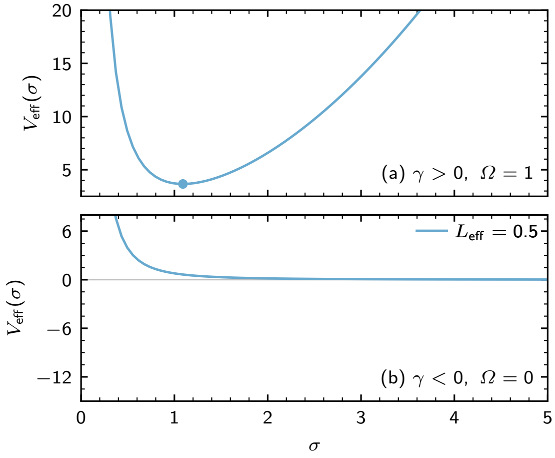
Annular modes ($p = 0$): one ring; σ sets its radius.

$$\xi_{pS}\,\ddot{\sigma} = -\frac{dV_{\mathrm{eff}}}{d\sigma}, \qquad V_{\mathrm{eff}}(\sigma) = \frac{\xi_{pS}}{2}\left(\frac{1}{\sigma^2} + \Omega^2\sigma^2\right) + \frac{\gamma\,|A|^2}{4\pi}\,\mathcal{I}_{pS}(\sigma).$$

$$\mathcal{I}_{pS}(\sigma) = \int d^2\mathbf{r}\;\rho_{pS}\,\Theta[\rho_{pS}] = 2\pi\int_0^\infty dq\;q\,\frac{L_{\mathrm{eff}}^2}{1+L_{\mathrm{eff}}^2 q^2}\;\tilde{\rho}_{pS}(q;\sigma)^2.$$

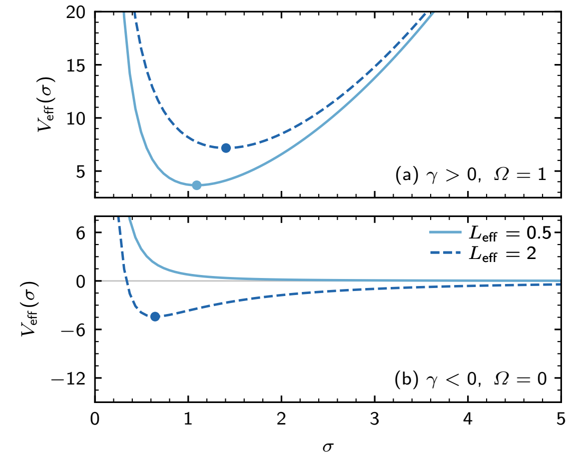
- Three Euler–Lagrange equations, for A, σ, b. The amplitude decouples, $|A|^2 = |A_0|^2 \mathrm{e}^{-\alpha_0 z}$, and the chirp is $b = \dot{\sigma}/\sigma$.
- What remains is **one** equation: a **particle of mass** ξ_{pS} in the well V_{eff}, which depends on z only through $|A|^2$.
- **Loss** enters **phenomenologically**, through the decay of $|A|^2$ in V_{eff} alone.
- $\mathcal{I}_{pS}$: the **nonlocal interaction energy** of the profile $\rho_{pS} = T_{pS}^2$; $\tilde{\rho}_{pS}$ its Hankel transform.
- One radial integral. For the annular modes $(p=0)$ it is in **closed form**: for $S=0$, $\mathcal{I}_{00} = -\pi\,\mathrm{e}^{u}\,\mathrm{Ei}(-u)$ with $u = \sigma^2/2L_{\mathrm{eff}}^2$.
- Local limit $\sigma \gg L_{\mathrm{eff}}$: $\mathcal{I}_{0S} \propto L_{\mathrm{eff}}^2/\sigma^2$, the **Kerr** scaling.

- $V_{\rm eff}$ for an annular mode with $S = 2$, at three nonlocal lengths; dots mark the minima.
- **(a) Defocusing in the trap.** The stationary width grows with $L_{\rm eff}$: the beam **broadens**.
- **(b) Focusing without trap.** Diffraction exceeds the logarithm as $\sigma \to 0$: **no collapse**.



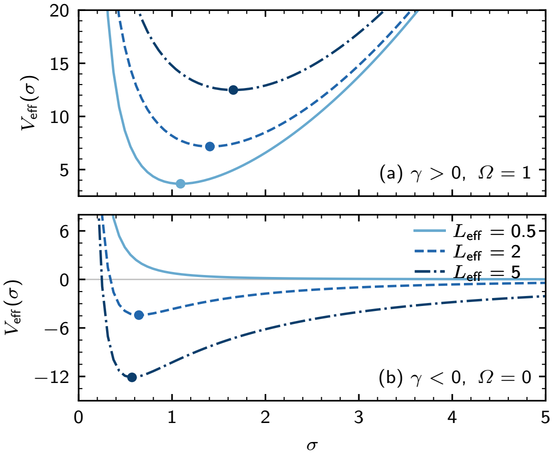
Stationary states: L_{eff} sets the width, and there is no collapse

- V_{eff} for an annular mode with $S = 2$, at three nonlocal lengths; dots mark the minima.
- **(a) Defocusing in the trap.** The stationary width grows with L_{eff}: the beam **broadens**.
- **(b) Focusing without trap.** Diffraction exceeds the logarithm as $\sigma \to 0$: **no collapse**.

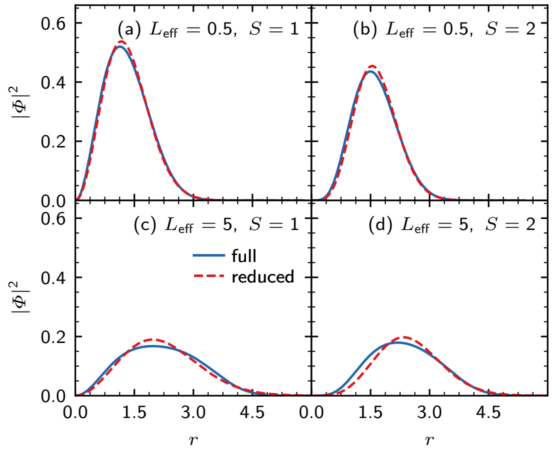


Stationary states: L_{eff} sets the width, and there is no collapse

- V_{eff} for an annular mode with $S = 2$, at three nonlocal lengths; dots mark the minima.
- **(a) Defocusing in the trap.** The stationary width grows with L_{eff}: the beam **broadens**.
- **(b) Focusing without trap.** Diffraction exceeds the logarithm as $\sigma \to 0$: **no collapse**.

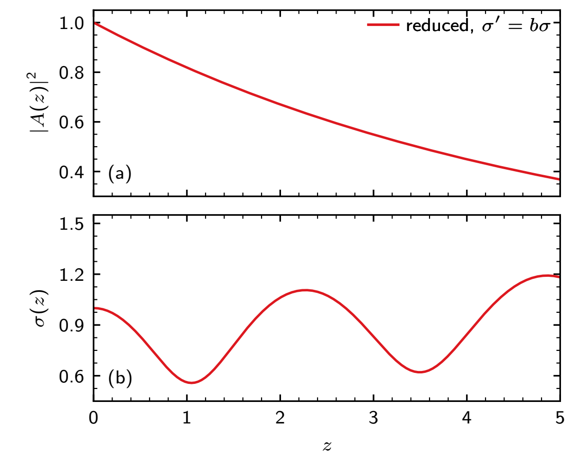


- **Full model**: imaginary time at fixed charge m. **Reduced**: the mode at the minimum of V_{eff}.

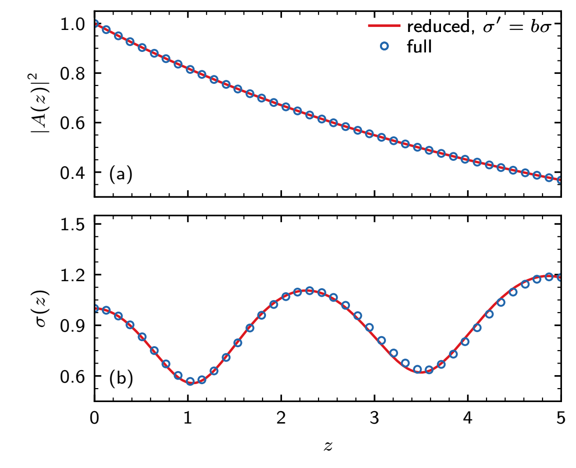


Propagation of a lossy vortex beam

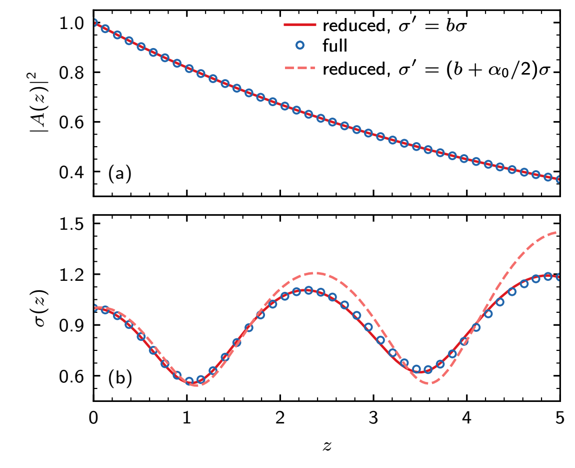
- An $m = 1$ vortex launched at the linear-mode width: focusing, $L_{\text{eff}} \gg \sigma$, lossy, trapped.
- **(b)** Loss in the amplitude alone: the reduced width follows the full run within 4%.
- Reference: a 1 W vortex ring in lead glass, $L_{\text{eff}}/w_0 \approx 40$ as here, $\alpha_0 \approx 0.015$, no trap.
  [C. Rotschild et al., Phys. Rev. Lett. **95**, 213904 (2005)]



- An $m = 1$ vortex launched at the linear-mode width: focusing, $L_{\mathrm{eff}} \gg \sigma$, lossy, trapped.
- **(b)** Loss in the amplitude alone: the reduced width follows the full run within 4%.
- Reference: a 1 W vortex ring in lead glass, $L_{\mathrm{eff}}/w_0 \approx 40$ as here, $\alpha_0 \approx 0.015$, no trap.
[C. Rotschild et al., Phys. Rev. Lett. **95**, 213904 (2005)]



- An $m = 1$ vortex launched at the linear-mode width: focusing, $L_{\mathrm{eff}} \gg \sigma$, lossy, trapped.
- **(b)** Loss in the amplitude alone: the reduced width follows the full run within 4%.
- Reference: a 1 W vortex ring in lead glass, $L_{\mathrm{eff}}/w_0 \approx 40$ as here, $\alpha_0 \approx 0.015$, no trap.
  [C. Rotschild et al., Phys. Rev. Lett. **95**, 213904 (2005)]



Conclusions

1. Thermal nonlocality between fixed-temperature facets, at weak absorption: a K_0 **kernel** of range $L_{\mathrm{eff}} = L/\pi$.

2. One annular mode with a chirp: a **reduced equation for the width**; the nonlocality enters through $\mathcal{I}_{pS}(\sigma)$ alone.

3. In the simulated regimes the reduced width **follows the full model closely**; azimuthal stability is not captured.

Based on

[FL et al., Nanophotonics 14, 805–813 (2025)]
[FL et al., Front. Phys. 21, 102203 (2026)]

Conclusions

1. Thermal nonlocality between fixed-temperature facets, at weak absorption: a K_0 **kernel** of range $L_{\text{eff}} = L/\pi$.
2. One annular mode with a chirp: a **reduced equation for the width**; the nonlocality enters through $\mathcal{I}_{pS}(\sigma)$ alone.
3. In the simulated regimes the reduced width **follows the full model closely**; azimuthal stability is not captured.

Based on

[FL et al., Nanophotonics 14, 805–813 (2025)]
[FL et al., Front. Phys. 21, 102203 (2026)]

Thank you for your attention

Questions?

Francesco Lorenzi, Luca Salasnich
francesco.lorenzi.2@phd.unipd.it

F. L. thanks M. Santagiustina and P. Villoresi for useful discussions.

UNIVERSITÀ DI PADOVA
Dipartimento di Ingegneria dell'Informazione

Dipartimento di Fisica e Astronomia "Galileo Galilei"

INFN

References

[1] D. Anderson, Phys. Rev. A **27**, 3135 (1983).

[2] B. A. Malomed, in *Progress in Optics*, Vol. 43 (Elsevier, 2002), pp. 71–193.

[3] V. M. Pérez-García et al., Phys. Rev. Lett. **77**, 5320 (1996).

[4] L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A **65**, 043614 (2002).

[5] F. Lorenzi and L. Salasnich, Nanophotonics **14**, 805 (2025).

[6] W. H. Renninger and F. W. Wise, Nat. Commun. **4**, 1719 (2013).

[7] O. Bang et al., Phys. Rev. E **66**, 046619 (2002).

[8] A. I. Yakimenko, Y. A. Zaliznyak, and Y. Kivshar, Phys. Rev. E **71**, 065603 (2005).

[9] C. Rotschild et al., Phys. Rev. Lett. **95**, 213904 (2005).

[10] C. Conti, M. Peccianti, and G. Assanto, Phys. Rev. Lett. **92**, 113902 (2004).

[11] A. Alberucci et al., Phys. Rev. A **91**, 013841 (2015).

[12] C. Conti, G. Ruocco, and S. Trillo, Phys. Rev. Lett. **95**, 183902 (2005).

[13] F. Lorenzi and L. Salasnich, Front. Phys. **21**, 102203 (2026).

[14] H. R. Lewis, Phys. Rev. Lett. **18**, 510 (1967).

[15] N. Ghofraniha et al., Phys. Rev. Lett. **99**, 043903 (2007).

References

[16] A. Alberucci et al., Phys. Rev. A **82**, 023806 (2010).

[17] G. Marcucci et al., Adv. Phys. X **4**, 1662733 (2019).

[18] W. Bao and Q. Du, SIAM J. Sci. Comput. **25**, 1674 (2004).

[19] W. Krolikowski et al., Phys. Rev. E **64**, 016612 (2001).

[20] D. Briedis et al., Opt. Express **13**, 435 (2005).

[21] M. Peccianti et al., Nature **432**, 733 (2004).

[22] Y. Izdebskaya et al., Nat. Commun. **8**, 14452 (2017).

[23] A. Piccardi et al., Appl. Phys. Lett. **96**, 071104 (2010).

[24] M. Kwaśny et al., Opt. Lett. **45**, 2451 (2020).

[25] U. A. Laudyn et al., Materials **11**, 1837 (2018).

[26] M. Peccianti and G. Assanto, Phys. Rep. **516**, 147 (2012).