

Nonlinear physics of ultracold quantum gases and optical fields in media

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Objectives of the thesis

- Study nonlinear wave phenomena across **ultracold atomic gases** and **nonlinear optics**, developing common techniques.
- Develop mathematical frameworks generalizing the Gross–Pitaevskii equation and Nonlinear Schrödinger equations (NLSE).
- Investigate the role of dimensionality, non-universal corrections from microscopic physics, and quantum fluctuations.

Presentation outline

- **Part 1: background and motivation:** Fundamental formalism, low-energy scattering theory, aspects of solitary waves for classical fields.
- **Part 2: results and developments:** Dimensional reduction techniques, soliton stability, and beyond-mean-field effects.

Part 1

Background and motivation

Functional integrals for Bosonic atoms: mean field and fluctuations

[L. Salasnich et al., *Phys. Rep.* 640, 1 (2016)]

The common starting point is the grand-canonical partition function

$$\mathcal{Z} = \int \mathcal{D}[\psi, \psi^*] e^{-S_E[\psi, \psi^*]/\hbar},$$

with Euclidean action

$$S_E = \int d\tau d^3\mathbf{r} \psi^* \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m} \nabla^2 - \mu + V_{\text{trap}} \right] \psi + \frac{g}{2} |\psi|^4.$$

Here $\psi(\mathbf{r}, \tau)$ is the Bose field, τ is imaginary time, m is the atomic mass, μ is the chemical potential, $V_{\text{trap}}(\mathbf{r})$ is the external potential, and g is the effective two-body interaction coefficient.

Two approaches organize the whole thesis:

- the saddle point $\delta S_E / \delta \psi^* = 0$, which yields the Gross–Pitaevskii description:

$$\mu \psi_0 = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} + g |\psi_0|^2 \right] \psi_0$$

- the Gaussian expansion $\psi = \psi_0 + \eta$, which, in the uniform case ($V_{\text{trap}} \equiv 0$) yields the Bogoliubov spectrum:

$$E_{\mathbf{q}} = \sqrt{\frac{\hbar^2 q^2}{2m} \left(\frac{\hbar^2 q^2}{2m} + 2gn_0 \right)}$$

Here ψ_0 is the condensate field, $\mathbf{q}$ is the excitation wave vector, and n_0 is the condensate density, and $g > 0$.

Scattering theory: low-energy input of the field theory

[H. A. Bethe, *Phys. Rev.* 76, 38–50 (1949)]
[E. Braaten et al., *Ann. Phys.* 323, 1770–1815 (2008)]

The many-body theory is fixed by a few **low-energy two-body observables**. For the s-wave channel one writes the scattering amplitude

$$f_0(k) = \frac{1}{k \cot \delta_0(k) - ik}$$

and it is possible to **expand** for low momentum

$$k \cot \delta_0(k) = -\frac{1}{a_s} + \frac{1}{2} r_{\text{eff}} k^2 + \dots,$$

where k is the relative wave number, $\delta_0(k)$ is the s-wave phase shift, a_s is the scattering length, and r_{eff} is the effective range. The same information is encoded in a **low-momentum interaction**

$$V(k) \simeq g + g_2 k^2 + \dots,$$

with g the universal contact coupling,

$$g = \frac{4\pi\hbar^2}{m} a_s$$

This reduction is meaningful given:

- only the s-wave channel is important in the **dilute** low energy limit (bosonic) case;
- the wavelength is long compared with the interaction range, so microscopic details enter only through a **few** scattering parameters;
- a_s controls the universal limit, while r_{eff} measures the first **nonuniversal** correction.

The idea is to solve the two-body problem, extract a_s , r_{eff} , $\dots$, and use them as the controlled input of the effective many-body description.

Universal versus nonuniversal behaviour, Monte Carlo numerical results

[V. Cikojević et al., *Phys. Rev. A* 99, 023618 (2019)]
[E. Braaten et al., *Phys. Rev. A* 63, 063609 (2001)]

The universal approximation works remarkably well in dilute gases, but **Diffusion Monte Carlo** calculations show that once the density increases, the equation of state departs from a description based on a_s alone.

In the figure, different microscopic models with the same scattering length but different effective range **give different energies!**

This is the first motivation:

- determine effective couplings beyond the universal term;
- understand when r_{eff} becomes observable;
- model not only the 3D scattering, but also scattering in **lower dimensions**

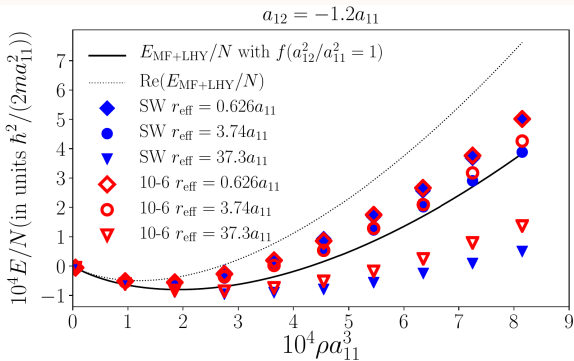


Figure adapted from V. Cikojević et al., *Phys. Rev. A* 99 (2019).

Ultracold regime: the three-dimensional Gross–Pitaevskii equation

[E. P. Gross, *Nuovo Cim.* 20, 454–477 (1961)]
[L. P. Pitaevskii, *Sov. Phys. JETP* 13, 451–454 (1961)]

At $T = 0$ and for dilute bosons, the many-body **dynamics** is reduced to the three-dimensional Gross–Pitaevskii equation,

$$i\hbar \frac{\partial}{\partial t} \psi = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(\mathbf{r}) + g|\psi|^2 \right] \psi.$$

Repulsive interactions correspond to $g > 0$, while **attractive** interactions correspond to $g < 0$.

Here $\psi(\mathbf{r}, t)$ is the condensate wavefunction, t is the real time.

In the situations of interest the confinement V_{trap} is typically anisotropic, composed by a strong transverse harmonic guide $V_{\perp}(x, y)$ plus an axial potential chosen for the problem at hand.

- **harmonic** axial confinement, $V_{\parallel}(z) = \frac{1}{2}m\omega_z^2 z^2$;
- narrow **barriers** and double-well Josephson traps, $V_{\parallel}(z) = V_0 e^{-z^2/(2\sigma^2)}$;
- **optical lattices**, $V_{\parallel}(z) = V_0 \sin^2(k_L z)$.

This is the natural starting point for discussing dimensional crossover: the central question is when the 3D-GPE **can be simplified** while retaining the relevant transverse physics.

The optical counterpart starts from manipulating Maxwell equations in matter:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2},$$

where $\mathbf{E}(\mathbf{r}, t)$ is the electric field, $\mathbf{P}(\mathbf{r}, t)$ is the polarization field, c is the vacuum speed of light, and μ_0 is the vacuum permeability.

In Kerr media one writes

$$\mathbf{P} = \mathbf{P}_L + \mathbf{P}_{NL}, \quad \mathbf{P}_{NL} \propto \chi^{(3)} |\mathbf{E}|^2 \mathbf{E},$$

where $\mathbf{P}_L$ is the linear polarization, $\mathbf{P}_{NL}$ is the nonlinear polarization, and $\chi^{(3)}$ is the third-order susceptibility.

The structure is the same as in ultracold gases: microscopic interactions are summarized into a coefficient controlling an **effective nonlinear wave equation**.

The Kerr reduction is relevant in the weakly nonlinear, nearly monochromatic regime:

- the response is **local** and effectively **instantaneous**;
- the cubic term is the leading correction to the linear dielectric response;
- a **scalar envelope** is adequate when polarization coupling and multimode effects stay weak.

Setting **effective external potentials** is almost as flexible in the case of matter-waves.

For a strongly confined transverse state (**quasi-1D**) one writes

$$\psi(\mathbf{r}, t) = f(z, t) \frac{1}{\sqrt{\pi} a_{\perp}} \exp\left(-\frac{x^2 + y^2}{2a_{\perp}^2}\right),$$

where $f(z, t)$ is the axial field and

$$a_{\perp} = \sqrt{\frac{\hbar}{m\omega_{\perp}}}$$

is the transverse oscillator length.

Integrating the transverse variables from the 3D-GPE leads to

$$i\hbar \frac{\partial}{\partial t} f = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + g_{1D} |f|^2 \right] f,$$

where $g_{1D} = g/(2\pi a_{\perp}^2)$ is the effective one-dimensional coupling.

This is **one of the simplest reduced models**. The thesis asks when it is reliable and how to improve it once the transverse width becomes dynamical. Remarkably, it supports **solitons** when $g_{1D} < 0$, but it becomes unreliable in some cases.

For stationary states one sets

$$f(z, t) = \phi(z) e^{-i\mu t/\hbar},$$

and for $g_{1D} < 0$ the reduced equation admits the bright-soliton profile

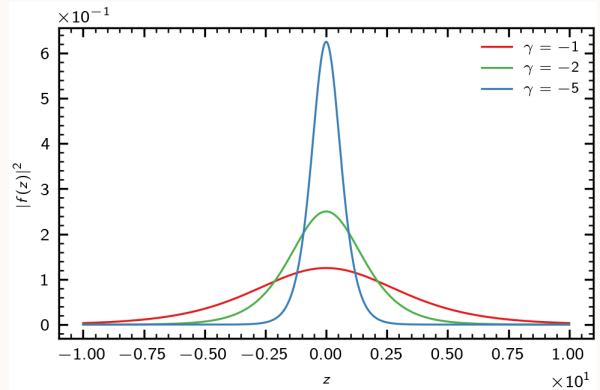
$$\phi(z) = \sqrt{\frac{2|\mu|}{|g_{1D}|}} \operatorname{sech}\left(\frac{\sqrt{2m|\mu|}}{\hbar} z\right),$$

where $\mu < 0$ is the chemical potential, and $\mu \propto -\gamma^2$. This can be understood as a result of balancing **kinetic term** and **attractive interaction**.

For $g_{1D} < 0$, the one-dimensional equation admits bright-soliton solutions. The balance is simple and fundamental:

- kinetic dispersion tends to broaden the wave packet;
- attractive interactions tend to compress it;
- at equilibrium the axial profile becomes localized.

This is the natural entry point to the soliton part of the thesis, but it is only the first approximation. In the real quasi-one-dimensional geometry, the transverse degree of freedom can return and change both stability and dynamics.



A similar simple reduction is available for optics, assuming **single-mode propagation** in a waveguide

$$\frac{\partial A}{\partial z} = -\frac{\alpha}{2}A - i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + i\gamma|A|^2A.$$

Here $A(z, T)$ is the optical envelope, z is the propagation coordinate, $T = t - z/v_g$ is the retarded time, α is the attenuation coefficient, β_2 is the group-velocity-dispersion parameter, γ is the Kerr nonlinearity coefficient, and v_g is the group velocity.

This equation is the optical analogue of the 1D-GPE:

- **chromatic dispersion** can balance **Kerr nonlinearity** eventually ($\gamma > 0$);
- localized pulses and solitons may emerge;

The physical regime is sharply encoded in the coefficients. Importantly:

- $\beta_2 < 0$ gives anomalous dispersion, where bright temporal solitons are possible;
- $\beta_2 > 0$ gives normal dispersion, where the same cubic term does not support the same localized balance;
- When the loss/gain term α is negligible, the equation can be determined from a **variational principle** in a direct way.

Why dimensional reduction?

Geometry, attraction, and collapse

[Y. Kagan et al., *Phys. Rev. Lett.* 81, 933 (1998)]
[L. Salasnich et al., *Phys. Rev. A* 65, 043614 (2002)]
[FL et al., *Sci. Rep.* 14, 4665 (2024)]

In elongated traps one would like to replace the 3D equation with a one-dimensional model. This is justified **only** if the transverse mode remains **fixed**.

The attractive regime is precisely where this simplification becomes delicate:

- bright solitons require $a_s < 0$;
- stationary solutions (solitons) may be **unstable**;
- the instability, when considered dynamically, make the wave localize more and more, until **effect appearing at higher densities start to dominate**.

Dimensional reduction is a step that **must be tested**.

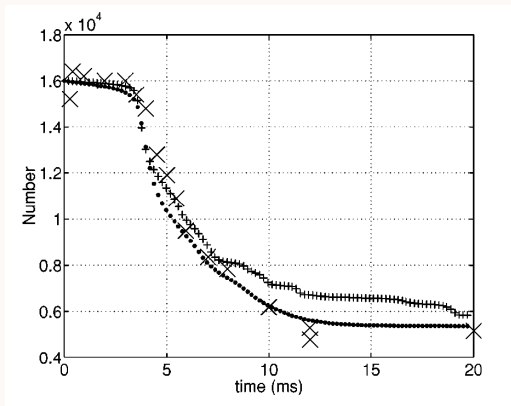


Figure adapted from [Y. Kagan et al., *Phys. Rev. Lett.* 81, 933 (1998)].

In general dimension D , the nonlinear Schrödinger equation is

$$i\frac{\partial}{\partial t}\psi + \nabla^2\psi + |\psi|^{2\sigma}\psi = 0,$$

where ψ is the envelope field, σ is the power of the nonlinearity, and ∇^2 is the D -dimensional Laplacian.

Its collapse threshold is fixed by

$$D_c = \frac{2}{\sigma}.$$

For the cubic case $\sigma = 1$, the critical dimension is $D_c = 2$: one-dimensional cubic solitons are stable, while in two and three dimensions self-focusing can drive blow-up.

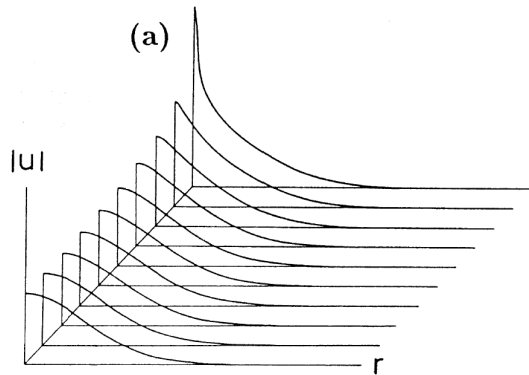


Figure adapted from [L. Bergé, *Phys. Rep.* 303, 259–370 (1998)].

If the **effective range** is retained, the mean-field dynamics is modified to

$$i\hbar \frac{\partial}{\partial t} \psi = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} + g|\psi|^2 + \frac{g_2}{2} \nabla^2 |\psi|^2 \right] \psi.$$

Here g is the contact coupling and g_2 is the coupling associated with the effective range.

The consequences already appear at mean-field (GPE) level, within a 1D reduction, for example soliton profiles change.

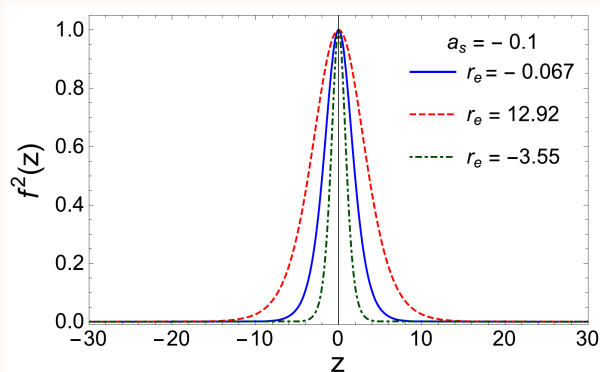


Figure adapted from [F. Sgarlata et al., *J. Phys. B: At. Mol. Opt. Phys.* 48, 115301 (2015)].

Applications of solitary waves: matter-wave interferometry

[J. L. Helm et al., *Phys. Rev. A* 89, 033610 (2014)]
[J. L. Helm et al., *Phys. Rev. Lett.* 114, 134101 (2015)]
[FL et al., *Sci. Rep.* 14, 4665 (2024)]

Bright solitons are interesting for **interferometry** because they remain localized over long propagation distances and can be split and recombined by narrow barriers.

The output populations N_L and N_R and the relative phase ϕ are then sensitive to external perturbations on **one interferometer arm**.

This application is also a source of non-ideal effects:

- **transverse excitation** during splitting;
- barrier-induced **dynamical collapse** in the attractive regime;

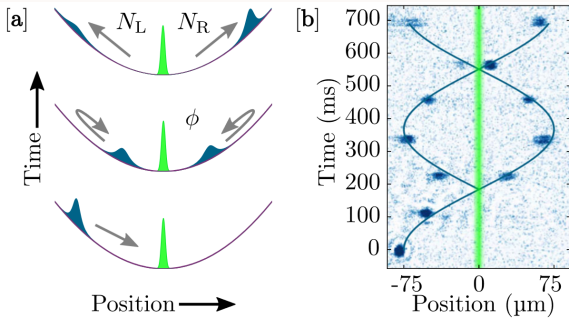
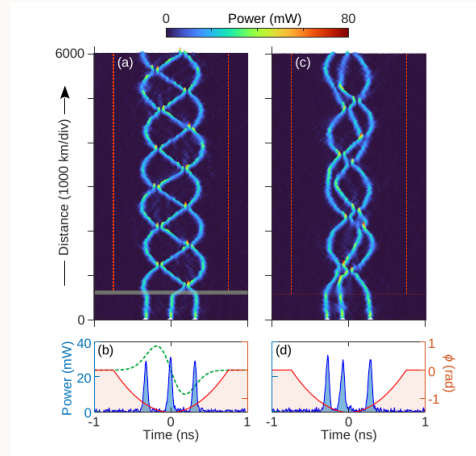
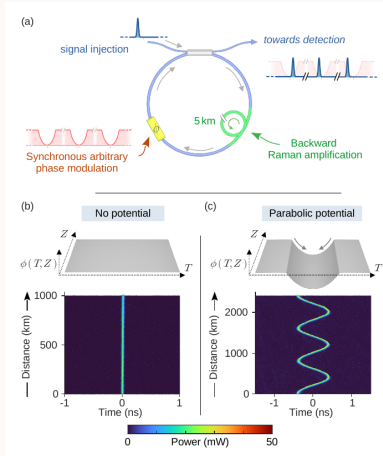


Figure adapted from [O. J. Wales et al., *Commun Phys* 3, 1–9 (2020)].

Applications of solitary waves: optical fiber loops

[P. Parra-Rivas et al., *Opt. Commun.* 546, 129749 (2023)]

[F. Copie et al., *Phys. Rev. Research* 7, 043232 (2025)]



Figures adapted from [F. Copie et al., *Phys. Rev. Research* 7, 043232 (2025)].

Atomic Josephson junctions: oscillations and self-trapping

[B. D. Josephson, *Physics Lett.* 1, 251 (1962)]
[A. Smerzi et al., *Phys. Rev. Lett.* 79, 4950–4953 (1997)]
[S. Raghavan et al., *Phys. Rev. A* 59, 620–633 (1999)]
[M. Albiez et al., *Phys. Rev. Lett.* 95, 010402 (2005)]

The GPE dynamics in a **double well** can be reduced into **two-modes** in terms of the population imbalance $z = (N_L - N_R)/N$ and relative phase ϕ :

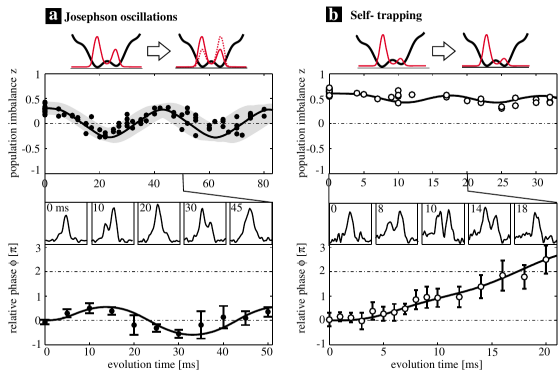
$$\frac{dz}{dt} = -\frac{J}{\hbar} \sqrt{1-z^2} \sin \phi, \quad \frac{d\phi}{dt} = \frac{UN}{\hbar} z + \frac{J}{\hbar} \frac{z \cos \phi}{\sqrt{1-z^2}}.$$

Here J is the tunneling energy, U is the onsite interaction coefficient, and N is the total atom number.

These equations predict two regimes:

- Josephson oscillations around $z = 0$;
- macroscopic quantum self-trapping, where the time average of z stays nonzero.

How are **quantum fluctuations** changing the dynamical properties of the junction?



From [M. Albiez et al., *Phys. Rev. Lett.* 95, 010402 (2005)] and
[S. Raghavan et al., *Phys. Rev. A* 59, 620–633 (1999)].

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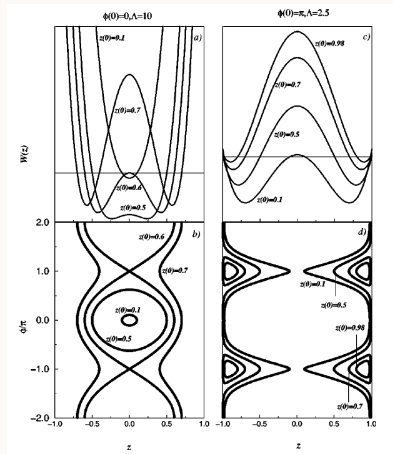
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[S. Raghavan et al., *Phys. Rev. A* 59, 620–633 (1999)].

Part 2

Results and developments

T-matrix formalism: Lippmann–Schwinger equation

[J. R. Taylor, Courier Corporation (2012)]
[R. G. Newton, Springer (2013)]
[FL et al., *Phys. Rev. A* 107, 033325 (2023)]

The two-body problem can be cast in the **Lippmann–Schwinger** equation

$$T(\mathbf{k}', \mathbf{k}; E) = V(\mathbf{k}', \mathbf{k}) + \int \frac{d^D \mathbf{q}}{(2\pi)^D} \frac{V(\mathbf{k}', \mathbf{q}) T(\mathbf{q}, \mathbf{k}; E)}{E - \hbar^2 q^2 / m + i\epsilon}.$$

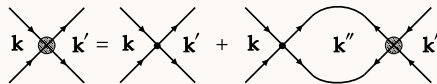
Here T is the transition matrix, V the interaction potential operator, $E = \hbar^2 k^2 / m$ the relative energy, and D the spatial dimension.

In three dimensions one expands in Legendre polynomials,

$$T(\mathbf{k}', \mathbf{k}; E) = \sum_{\ell=0}^{\infty} (2\ell + 1) T_{\ell}(k) P_{\ell}(\cos \theta),$$

where P_{ℓ} is the Legendre polynomial of order ℓ and θ is the scattering angle (can be generalized in D dimensions!).

We focus on the *s-wave* channel $\ell = 0$, because it fixes the low-energy couplings of the bosonic field theory.



T-matrix formalism: on-shell reduction in 1D and 3D

[H. A. Bethe, *Phys. Rev.* 76, 38–50 (1949)]
[FL et al., *Phys. Rev. A* 107, 033325 (2023)]
[FL et al., *Ann. Phys.*, e00227 (2025)]

For the s-wave component one writes

$$T_0(k) = V_0(k) + S_D \int_0^\infty \frac{dq}{(2\pi)^D} \frac{q^{D-1}}{\hbar^2 k^2/m - \hbar^2 q^2/m + i\epsilon} V_0(k, q) T_0(q, k).$$

Here S_D is the solid angle in D dimensions, while V_0 and T_0 are the s-wave interaction kernel and s-wave T matrix.

The **on-shell approximation** replaces the matrices in the integral by

$$V_0(k, q) \approx V_0(k), \quad T_0(q, k) \approx T_0(k),$$

so that

$$T_0(k) = \left(\frac{1}{V_0(k)} - C_D(k) \right)^{-1}, \quad C_D(k) = -\frac{m}{\hbar^2} (-ik)^{D-2} \frac{\Gamma(1 - D/2)}{(4\pi)^{D/2}}.$$

Matching with the phase shift $\delta_0(k)$ gives

$$V_{0,1D}(k) = -\frac{2\hbar^2}{m} k \tan \delta_0(k), \quad V_{0,3D}(k) = \frac{4\pi\hbar^2}{m} \frac{\tan \delta_0(k)}{k}.$$

Two dimensions are special because the loop integral is logarithmic:

$$C_{2D}(k) = \frac{m}{2\pi\hbar^2} \ln\left(\frac{k}{2} \frac{e^{\gamma_E/2}}{\Lambda}\right) - i \frac{m}{4\hbar^2}.$$

Here γ_E is the Euler–Mascheroni constant and Λ is the ultraviolet momentum scale introduced by regularization. The on-shell interaction then becomes

$$V_{0,2D}(k) = -\frac{4\hbar^2}{m} \frac{1}{\cot \delta_0(k) - \frac{2}{\pi} \ln\left(\frac{k}{2} \frac{e^{\gamma_E/2}}{\Lambda}\right)}.$$

This logarithm is why the low-energy expansion in two dimensions differs qualitatively from both one and three dimensions: the scattering scale enters through a renormalized logarithm rather than a simple power series in k .

The approximation is reliable when the product of the potential and T -matrix elements in the integral of the Lippman-Schwinger equation **vary smoothly**.

In practice this means:

- energy well separated from s -wave bound states energies, which would make the kernel rapidly varying due to **divergence of the scattering length**;
- **weak potentials**, where the s -wave phase shift still captures the relevant low-energy information.

Exact solutions of the scattering problem are available in **all dimensions** for simple enough potentials like

$$V(r) = \gamma \delta^{(1)}(R - r),$$

and

$$V(r) = \begin{cases} \gamma, & r \leq R, \\ 0, & r > R, \end{cases}$$

In those cases, it is verified that the on-shell approximation provides a phase shift matching with the exact one **to leading order** in the low γ limit. So the method is a compact, dimension-aware map from the expansion of $\delta_0(k)$ to effective couplings.

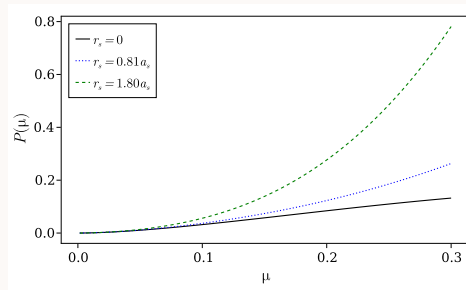
The two-body analysis feeds directly into the many-body equation of state. For the local effective Lagrangian density

$$\mathcal{L} = \psi^* \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 \right] \psi - \frac{\tilde{g}}{2} |\psi|^4 + \frac{\tilde{g}_2}{2} |\psi|^2 \nabla^2 |\psi|^2,$$

the one-loop pressure in two dimensions is then

$$P(\mu) = \frac{m}{8\pi\hbar^2} \mu^2 \ln \left(\frac{4\hbar^2}{m\mu a_s^2 e^{2\gamma_E + 1/2}} \right) + \frac{3m^2}{16\hbar^4} r_{\text{eff}}^2 \mu^3.$$

The first term is **universal**, while the second isolates the **leading finite-range correction**.



Using

$$\psi(\mathbf{r}, t) = f(z, t) \frac{e^{-(x^2+y^2)/(2\sigma^2)}}{\sqrt{\pi} \sigma},$$

and **allowing the transverse width to depend on z and t** , $\sigma(z, t)$ gives the following Euler-Lagrange equations for the two dynamical variables

$$i\hbar\partial_t f = \left[-\frac{\hbar^2}{2m}\partial_z^2 + V + \frac{\hbar^2}{2m\sigma^2}(1 + (\partial_z\sigma)^2) + \frac{m\omega_\perp^2}{2}\sigma^2 + \frac{2\hbar^2 a_s(N-1)}{m\sigma^2}|f|^2 \right] f,$$

$$\sigma^4 - a_\perp^4 [1 + 2a_s(N-1)|f|^2] + a_\perp^4 [\sigma\partial_z^2\sigma - (\partial_z\sigma)^2 + \sigma\partial_z\sigma\partial_z\ln|f|^2] = 0.$$

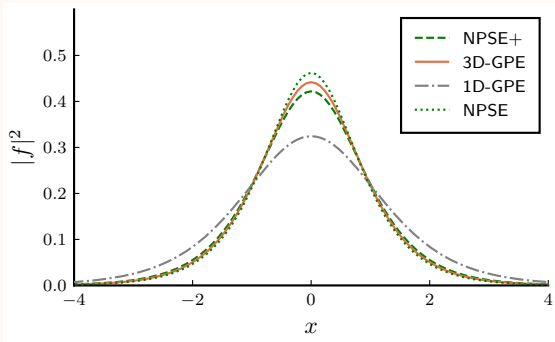
Neglecting derivative terms recovers the **Non Polynomial Schrödinger equation (NPSE)** closure

$$\sigma^2 = a_\perp^2 \sqrt{1 + 2a_s(N-1)|f|^2}.$$

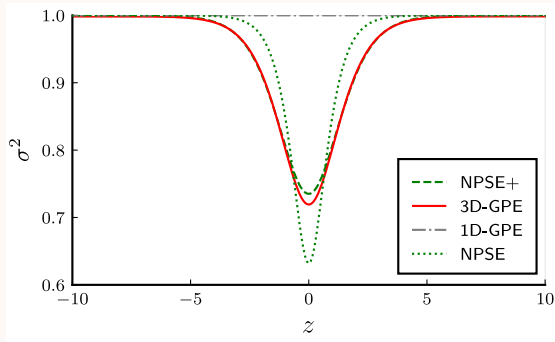
This is the key distinction: NPSE retains the local transverse response, while NPSE+ also keeps the longitudinal derivatives of the transverse width.

Solitary waves beyond 1D GPE: NPSE and NPSE+ near collapse

[L. Salasnich et al., *Phys. Rev. A* 65, 043614 (2002)]
[L. Salasnich et al., *Phys. Rev. A* 76, 063614 (2007)]
[FL et al., *Sci. Rep.* 14, 4665 (2024)]



Axial density profile



Transverse width

Near the static-collapse threshold, NPSE+ tracks both the axial profile and the transverse squeezing more accurately than the plain 1D cubic model.

A stringent dynamical test is the collision against a narrow Gaussian barrier

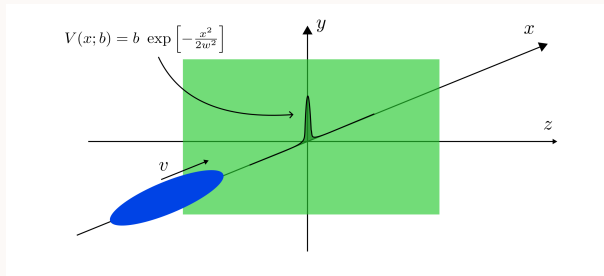
$$V(z; b) = b \exp\left(-\frac{z^2}{2w^2}\right),$$

The observable is the transmitted fraction

$$T = \int_0^\infty |f(z, t_{\text{fin}})|^2 dz,$$

where b is the barrier height and t_{fin} is the final time.

This problem, as a dynamical problem, is more demanding than the stationary one, and also particularly relevant for applications.



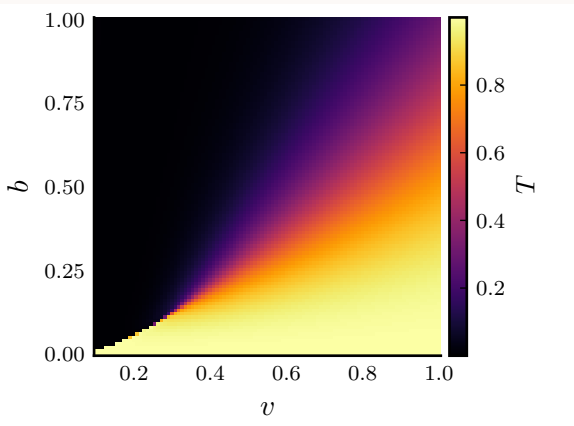
The reduced-model transmission diagram **becomes discontinuous** at low velocity and barrier height.

The reason is energetic:

$$E_k > E_G(N - n) + E_G(n) - E_G(N),$$

where E_k is the incoming kinetic energy and $E_G(n)$ is the ground-state energy of a fragment with n atoms.

Not every split is allowed. Also, using higher order models (3D-GPE), a dynamical collapse region is found.



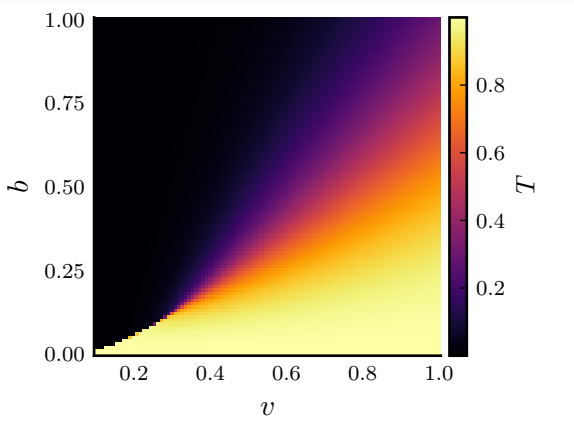
The reduced-model transmission diagram **becomes discontinuous** at low velocity and barrier height.

The reason is energetic:

$$E_k > E_G(N - n) + E_G(n) - E_G(N),$$

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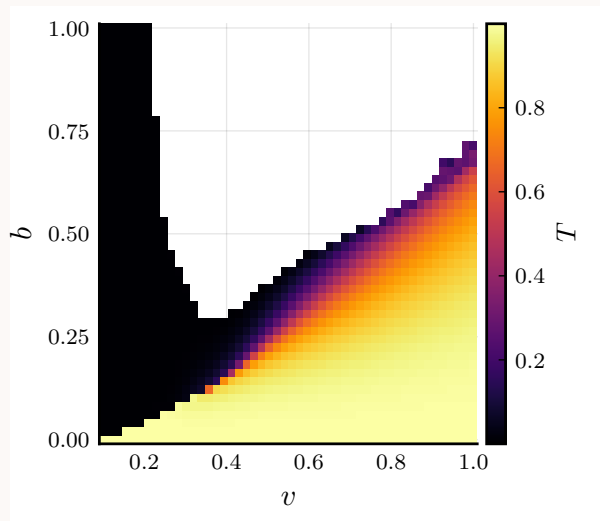
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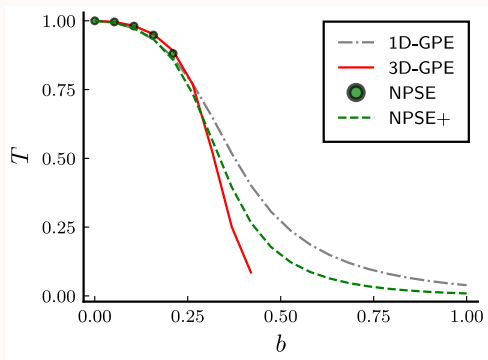
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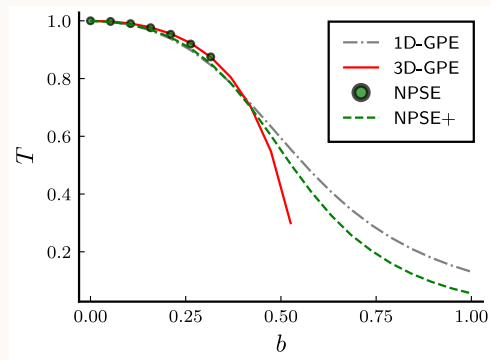
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Not every split is allowed. Also, using higher order models (3D-GPE), a dynamical collapse region is found.





$v = 0.6$



$v = 0.8$

The simplest way to stationary properties: Gaussian *ansatz*

[L. Salasnich et al., *Phys. Rev. A* 65, 043614 (2002)]
[Y. Kagan et al., *Phys. Rev. Lett.* 81, 933 (1998)]
[FL et al., *Sci. Rep.* 14, 4665 (2024)]

For attractive bosons in a cigar-shaped trap, a simple variational ansatz is

$$\psi(\mathbf{r}) = \frac{1}{\pi^{3/4} \sigma \eta^{1/2}} \exp \left[-\frac{x^2 + y^2}{2\sigma^2} - \frac{z^2}{2\eta^2} \right].$$

Here σ is the **transverse** Gaussian **width** and η is the **axial width**.

The corresponding reduced energy reads

$$E(\eta, \sigma) = \frac{1}{2} \left(\frac{1}{2\eta^2} + \frac{1}{\sigma^2} + \sigma^2 \right) + \frac{g}{2\sqrt{2\pi}} \frac{1}{\sigma^2 \eta},$$

where $g = 2(N - 1)a_s/a_\perp$ is the dimensionless interaction strength and $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$ is the transverse oscillator length.

This baseline already captures the main quasi-one-dimensional mechanism: axial self-focusing squeezes the transverse cloud, so the collapse threshold is an intrinsically three-dimensional effect.

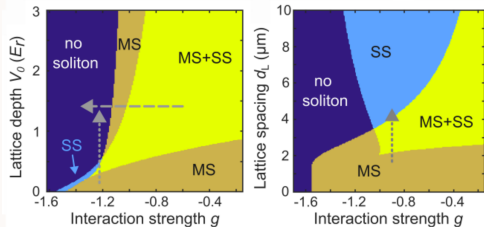
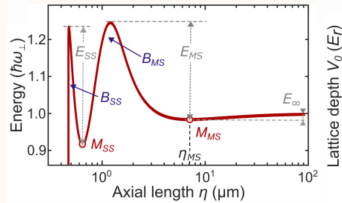
In an **optical lattice**, the Gaussian energy becomes

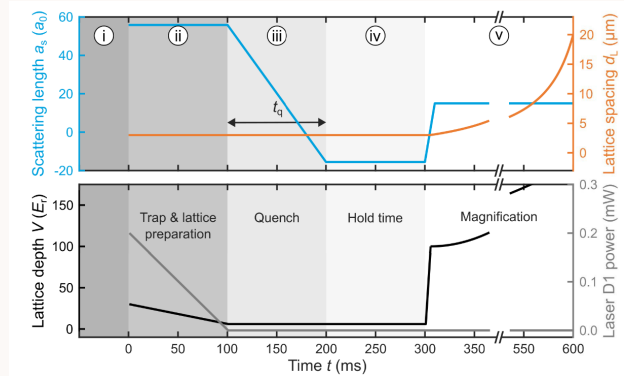
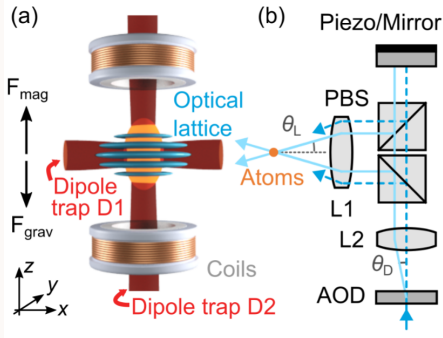
$$E = \frac{1}{2} \left(\frac{1}{2\eta^2} + \frac{1}{\sigma^2} + \sigma^2 \right) + \frac{g}{2\sqrt{2\pi}} \frac{1}{\sigma^2 \eta} - V_0 e^{-k_L^2 \eta^2}.$$

Here V_0 is the lattice depth and $k_L = \pi/d_L$ is the lattice wave number of the lattice potential $V = V_0 \cos(2k_L z)$. Recoil energy scale is $E_r = (\hbar\pi/d_L)^2/(2m)$.

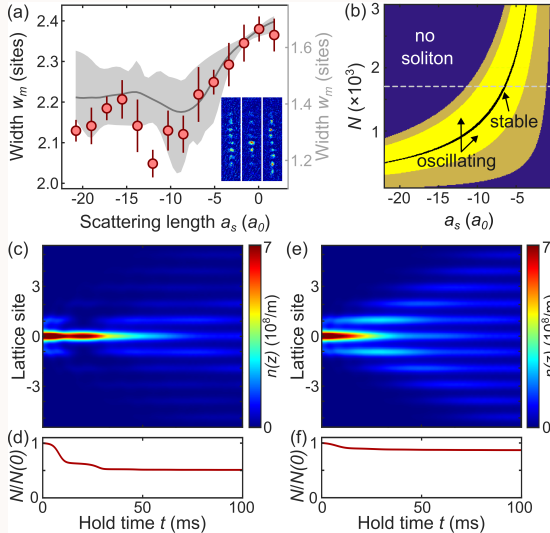
The lattice term **generates new local minima** corresponding to localized wave packets spread over one or several sites. The phase diagram organizes the main regimes:

- stable **single-site** or **multi-site localization** when a local minimum is protected by an energy barrier;
- combined presence of potential minima associated with single and multi-site solutions.
- **collapse** when the transverse confinement can no longer balance the axial focusing.



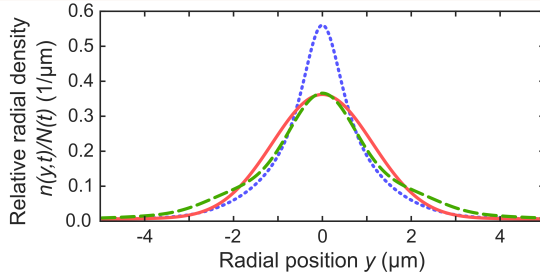


The experiment uses an ^{133}Cs condensate in a trap with accordion optical lattice. The key control knobs are the interaction (Feshbach resonance tuning) and the optical lattice depth and period.



Strong attraction drives the condensate to a high localization where **three-body losses** dominate.

- each dynamical collapse event is accompanied by **atom loss**, which prevents immediate irreversible blow-up;
- the overall sequence is captured only when the model includes full 3D dynamics together with a **phenomenological quintic loss term**.



The post-quench dynamics in strong attraction is triggering the a **transverse response** of the field.

- transverse squeezing changes both the onset time and the intensity of the collapse episodes;
- the density redistribution after each collapse depends sensitively on the transverse dynamics;
- purely one-dimensional models miss these trends, while NPSE-based reductions provide a more faithful intermediate description.

Solitary waves in graded index optical fibers. Extending the NPSE to Laguerre–Gauss modes

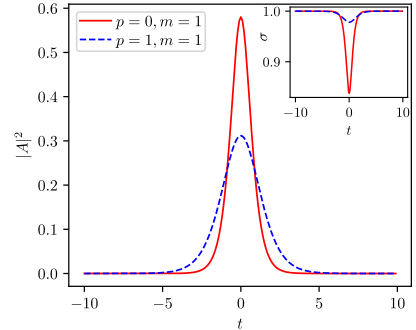
[FL et al., *Nanophotonics* 14, 805–813 (2025)]
[P. Parra-Rivas et al., *Opt. Commun.* 546, 129749 (2023)]

In a graded-index multimode fiber, the envelope obeys a $(3 + 1)$ -dimensional nonlinear Schrödinger equation. Using **Laguerre–Gauss modes** with radial index p and azimuthal index $S = |m|$ yields the reduced equation

$$i \frac{\partial A}{\partial z} = -\frac{1}{2} \frac{\partial^2 A}{\partial t^2} + \xi_{pS} \left(\frac{1 + \frac{3}{2} g_{pS} |A|^2}{\sqrt{1 + g_{pS} |A|^2}} \right) A,$$

with $\xi_{pS} = S + 2p + 1$.

Here $A(z, t)$ is the reduced axial envelope and g_{pS} is the effective nonlinear coupling for the chosen Laguerre–Gauss mode.



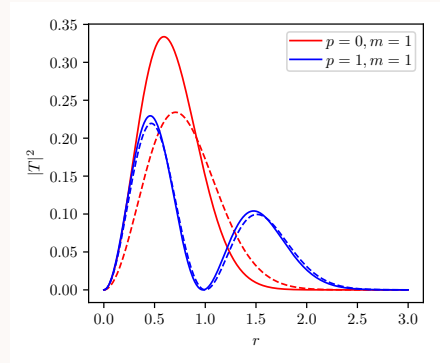
The stationary solutions, i.e. $A(z, t) = a(t) \exp[-i\kappa z]$, have an energy that can be written as

$$E = \frac{2\sqrt{2}}{3\gamma_{pS}}(2\kappa_{pS} + 1)\sqrt{1 - \kappa_{pS}},$$

where κ_{pS} is the variational width parameter and γ_{pS} is the effective nonlinear coefficient.

The **Vakhitov–Kolokolov** criterion ($\partial E / \partial \kappa_{pS} > 0$) selects the **stable branch** and identifies the collapse threshold.

Higher-order Laguerre–Gauss modes redistribute the energy between radial and axial directions, modifying both profile shape and stability window.

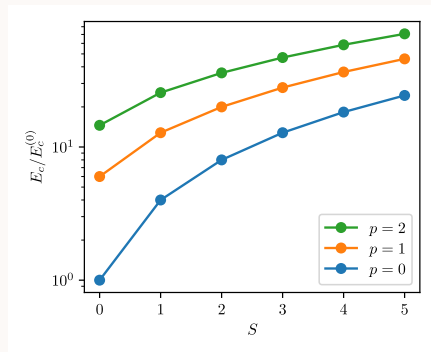


The generalized NPSE predicts the critical energy

$$E_c = \frac{4}{3\gamma_{pS}}.$$

Expanding the nonpolynomial interaction gives a **hierarchy of reduced models**: the **cubic** reduction never collapses, the **cubic–quintic** reduction collapses too late, and the full nonpolynomial theory gives the sharpest threshold.

This makes the Laguerre–Gauss extension a clean testing condition for comparing different effective nonlinearities beyond the fundamental transverse mode.



Beyond mean-field Josephson dynamics: effective two-site model

[A. Smerzi et al., *Phys. Rev. Lett.* 79, 4950–4953 (1997)]
[S. Raghavan et al., *Phys. Rev. A* 59, 620–633 (1999)]
[A. Bardin et al., *New J. Phys.* 26, 013021 (2024)]

For a two-site bosonic junction, the reduced Lagrangian can be written as

$$\mathcal{L} = \frac{N\hbar}{2} z \frac{d\phi}{dt} + \frac{JN}{2} \sqrt{1 - z^2} \cos \phi - E_{\text{int}}^{(D)}(z),$$

where z is the population imbalance, ϕ is the phase difference, J is the tunneling amplitude, and $E_{\text{int}}^{(D)}$ contains the **dimension-dependent** interaction energy.

Beyond mean field, the interaction term **is corrected** by the fluctuation energy:

$$\mathcal{E}_{1D} = \frac{1}{2}gn^2 - \frac{2}{3\pi} \sqrt{\frac{m}{\hbar^2}} (gn)^{3/2}, \quad \mathcal{E}_{3D} = \frac{1}{2}gn^2 + \frac{8}{15\pi^2} \left(\frac{m}{\hbar^2}\right)^{3/2} (gn)^{5/2},$$

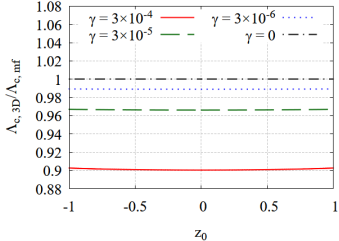
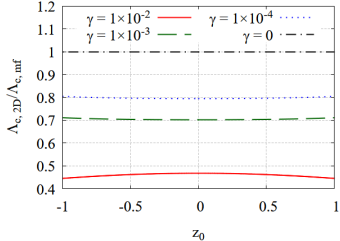
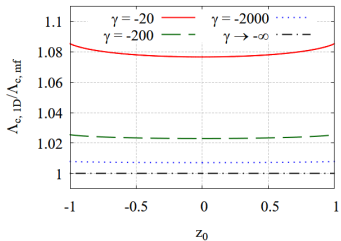
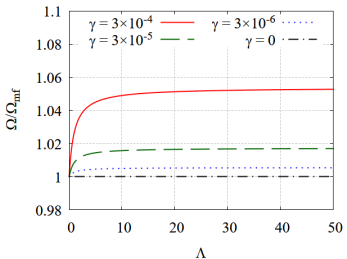
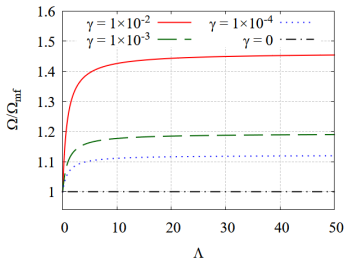
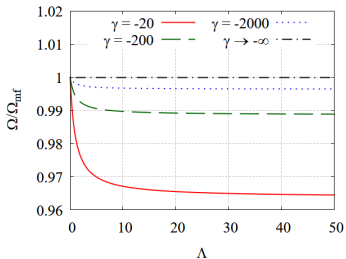
while in two dimensions the same mean-field is adopted, but with a modified coupling,

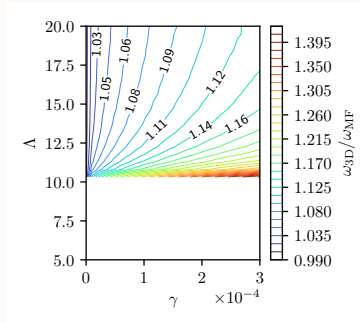
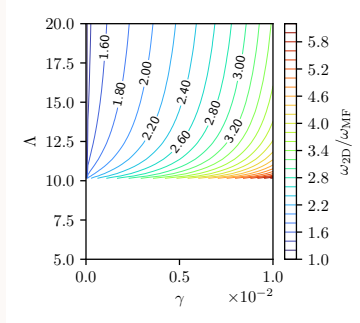
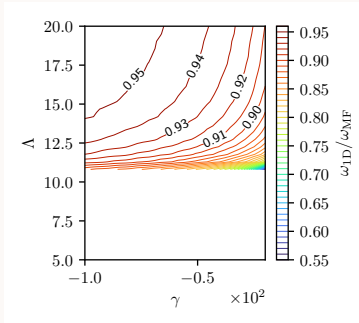
$$\mathcal{E}_{2D} = \frac{1}{2}g_r n^2, \quad g_r = -\frac{4\pi\hbar^2}{m} \frac{1}{\ln(Cn)}, \quad C = \pi e^{2\gamma_E + 1} a_s^2.$$

Linearization of the EL equations around $z = 0$ and $\phi = 0$ gives a **corrected Josephson frequency** Ω_D and, away from the linear regime, a **shifted self-trapping threshold** $\Lambda_{c,D}$.

Beyond mean-field Josephson dynamics: frequency and MQST

[A. Bardin et al., *New J. Phys.* 26, 013021 (2024)]





In the self-trapped state the **phase-slippage rate** is

$$\omega = \left\langle \frac{d\phi}{dt} \right\rangle = \frac{\langle \Delta\mu \rangle}{\hbar},$$

where $\Delta\mu$ is the instantaneous chemical-potential difference between the two wells.

The beyond-mean-field correction reduces ω in one dimension, enhances it in three dimensions, and is largest in two dimensions, mirroring the trend already seen in the Josephson frequency.

Starting from the density-phase action

$$S[n, \theta] = \int dt d^D \mathbf{r} \left[-\mathcal{E}_0(n) - n\hbar \frac{\partial \theta}{\partial t} - n \frac{\hbar^2}{2m} |\nabla \theta|^2 + \mu n \right],$$

the phase enters as a shifted chemical potential

$$\mu_{\text{eff}} = \mu - \hbar \frac{\partial \theta}{\partial t} - \frac{\hbar^2}{2m} |\nabla \theta|^2.$$

Integrating out the density gives the exact only-phase Popov Lagrangian density, which can be expanded

$$\tilde{\mathcal{L}} = P(\mu_{\text{eff}}) = \tilde{\mathcal{L}} = P(\mu) + n \left(\hbar \partial_t \theta + \frac{\hbar^2}{2m} |\nabla \theta|^2 \right) + \frac{1}{2} \frac{n}{mc_s^2} \left(\hbar \partial_t \theta + \frac{\hbar^2}{2m} |\nabla \theta|^2 \right)^2 + \dots$$

This approach is an effective theory for zero-temperature BCS superconductors. It is possible to perform the calculations with minimal coupling to an electromagnetic field: EL equations are the Maxwell equations coupled to the superconducting field.

Electric and magnetic field static penetration lengths

[A. Peronio et al., *Phys. Rev. B* 94, 094503 (2016)]

Solving the EL equations, we get the penetration lengths

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda_L^2} \mathbf{B}, \quad B(x) = B(0)e^{-x/\lambda_L},$$

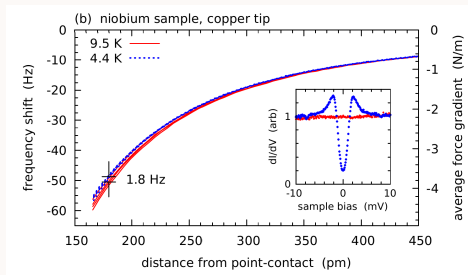
$$\lambda_L = \sqrt{\frac{m}{q^2 \bar{n}_s \mu_0}}.$$

$$\nabla^2 \mathbf{E} = \frac{1}{\lambda_E^2} \mathbf{E}, \quad E(x) = E(0)e^{-x/\lambda_E},$$

with

$$\lambda_E = \lambda_L \frac{c_s}{c}.$$

Typically $c \gg c_s$. **Experimental verification** of the penetration depth is of **formidable difficulty**, but a verification of the correct value would be interesting to assess the capabilities of the theory.



The work presented in the thesis may be summarized as follows:

- ➊ **Low-energy scattering** can be treated in a unified approximated way **across dimensions**, and the resulting formulas can be carried forward to effective many-body quantities.
- ➋ **Reduced nonlinear wave equations** are genuinely useful only in controlled regimes, failing especially near collapse. They are an important tool as they describe the **physics of solitary waves** across **different platforms**, nonlinear (Kerr) guided-wave optics and ultracold Bosons with attractive interaction being great examples.
- ➌ **Beyond-mean-field corrections** and phase-only actions extend the same effective-theory logic in an **analytically treatable way**, predicting simple corrections to the mean field statical properties and dynamics.

Natural continuations of this work include:

- Monte Carlo study (DMC) for the **stability against collapse** of the trapped Bose gas in **nonuniversal regime**: there are some preliminary results obtained with prof. Massignan and prof. Astrakharchik of UPC, Barcelona;
- Widely different kind of nonlinear interaction may be studied in the case of optical nonlinear media. A great example is the nonlocal **cubic nonlocal nonlinearity** mediated by the **thermo-optical effect**. A study titled "Optical nonlinear beams with thermal nonlocal interactions", in collaboration with prof. Salasnich, is under review in Frontiers of Physics.
- The effect of **finite temperature** in the dynamics of the **atomic Josephson junction** can be performed using a similar formalism to the one used for quantum fluctuations. A study done in collaboration with dr. Bardin and prof. Salasnich is in preparation for the submission.

Thank you for your attention!

This presentation is based on the following articles:

- FL, A. Bardin, and L. Salasnich, *On-shell approximation for the s-wave scattering theory*, Phys. Rev. A **107**, 033325 (2023).
- A. Bardin, FL, and L. Salasnich, *Quantum fluctuations in atomic Josephson junctions: the role of dimensionality*, New J. Phys. **26**, 013021 (2024).
- FL and L. Salasnich, *Atomic soliton transmission and induced collapse in scattering from a narrow barrier*, Sci. Rep. **14**, 4665 (2024).
- L. Salasnich, M. G. Pelizzo, and FL, *Only-phase Popov action: thermodynamic derivation and superconducting electrodynamics*, J. Phys. A: Math. Th. **57**, 355302 (2024).
- FL and L. Salasnich, *Variational approach to multimode nonlinear optical fibers*, Nanophotonics **14**, 805–813 (2025).
- FL and L. Salasnich, *Low-Energy Atomic Scattering: S-Wave Relation Between the Interaction Potential and the Phase Shift*, Ann. Phys. **537**, e00227 (2025).
- R. Cruickshank, FL, A. La Rooij, E. F. Kerr, T. Hilker, S. Kuhr, L. Salasnich, and E. Haller, *Experimental Observation of Single- and Multisite Matter-Wave Solitons in an Optical Accordion Lattice*, Phys. Rev. Lett. **135**, 263404 (2025).

Part A

Additional material

Scattering states, amplitudes, and cross sections

[H. A. Bethe, *Phys. Rev.* 76, 38–50 (1949)]
[R. G. Newton, Springer (2013)]
[E. Braaten et al., *Ann. Phys.* 323, 1770–1815 (2008)]

For two particles interacting through a central potential, the relative-motion problem is

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \frac{\hat{\mathbf{p}}^2}{2m_r},$$

and the scattering state is written asymptotically as

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta, k) \frac{e^{ikr}}{r}.$$

The differential cross section is fixed by the amplitude,

$$\frac{d\sigma}{d\Omega} = |f(\theta, k)|^2,$$

while in the zero-momentum limit one defines the scattering length through

$$\psi(\mathbf{r}) \simeq 1 - \frac{a_s}{r}, \quad \sigma \simeq 4\pi a_s^2.$$

Here m_r is the reduced mass, $\mathbf{k}$ the incoming relative wave vector, $f(\theta, k)$ the scattering amplitude, and a_s the scattering length.

For a central interaction one decomposes the wavefunction into partial waves,

$$\psi(\mathbf{r}) = \sum_{l=0}^{\infty} A_l P_l(\cos \theta) R_{kl}(r),$$

with asymptotic radial behavior

$$R_{kl}(r) \sim \frac{1}{kr} \sin\left(kr - \frac{\pi l}{2} + \delta_l\right).$$

The scattering amplitude then becomes

$$f(\theta, k) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos \theta), \quad f_l(k) = \frac{1}{k \cot \delta_l - ik},$$

and the total cross section is

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l.$$

For short-range potentials $\delta_l \sim k^{2l+1}$ at low momentum, so the $l=0$ channel dominates and

$$a_s = \lim_{k \rightarrow 0} \left(-\frac{\tan \delta_0}{k} \right).$$

Here P_l are Legendre polynomials, R_{kl} are radial wavefunctions, and δ_l are the phase shifts.

The exact scattering state may be rewritten as

$$|\psi^+\rangle = |\phi\rangle + \hat{G}_0^+ \hat{V} |\psi^+\rangle, \quad \hat{G}_0^+ = (E - \hat{H}_0 + i\varepsilon)^{-1},$$

which is the outgoing-wave Lippmann–Schwinger equation.

Introducing the transition operator through

$$\hat{T}|\phi\rangle = \hat{V}|\psi^+\rangle,$$

one obtains the operator identity

$$\hat{T} = \hat{V} + \hat{V} \hat{G}_0^+ \hat{T},$$

and therefore the Born series

$$\hat{T} = \hat{V} + \hat{V} \hat{G}_0^+ \hat{V} + \hat{V} (\hat{G}_0^+ \hat{V})^2 + \dots.$$

The observable scattering amplitude follows from the matrix element

$$f(\mathbf{k}', \mathbf{k}) = -\frac{m}{4\pi\hbar^2} T_{\mathbf{k}'\mathbf{k}}.$$

Here $|\phi\rangle$ is the incoming free state, $\hat{G}_0^+$ the free resolvent, and $T_{\mathbf{k}'\mathbf{k}} = \langle \mathbf{k}' | \hat{T} | \mathbf{k} \rangle$.

Why the ultracold regime is universal

[A. Leggett, Oxford Univ. Press (2006)]
[L. P. Pitaevskii et al., Clarendon Press (2003)]
[S. Giorgini et al., *Phys. Rev. A* 60, 5129 (1999)]

In dilute atomic gases the interparticle spacing is much larger than the microscopic range of the interaction, so two-body low-energy scattering carries the relevant information.

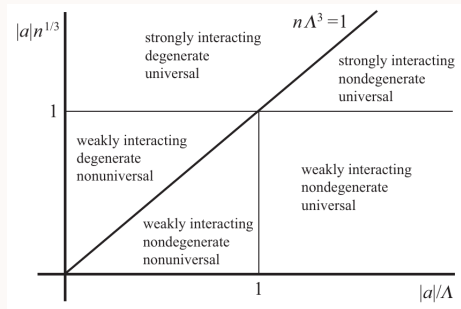
A realistic interaction has a van der Waals tail,

$$V(r) = -\frac{C_6}{r^6} - \frac{C_8}{r^8} - \frac{C_{10}}{r^{10}},$$

but in the ultracold limit its leading effect is compressed into a single parameter,

$$V_{\text{eff}}(\mathbf{r}) = g\delta^{(3)}(\mathbf{r}), \quad g = \frac{4\pi\hbar^2}{m}a_s.$$

Here C_6 , C_8 , and C_{10} are van der Waals coefficients, and the effective coupling is fixed entirely by the s-wave scattering length a_s inside the universal window.



Where the universal contact picture starts to fail

[H. T. C. Stoof et al., Springer (2009)]
[S. Giorgini et al., *Phys. Rev. A* 60, 5129 (1999)]
[FL et al., *Ann. Phys.*, e00227 (2025)]

The contact interaction is the leading low-energy description, but tunable systems can probe the first nonuniversal correction.

In the scattering language, one keeps the next term in the s-wave phase shift,

$$k \cot \delta_0(k) = -\frac{1}{a_s} + \frac{1}{2}r_{\text{eff}}k^2,$$

while in the effective-theory language one writes

$$V(k) \simeq g + g_2 k^2.$$

The parameter r_{eff} is the effective range, and g_2 is the coupling that reproduces its effect in the low-momentum expansion.

This is the regime reached, for instance, near a Feshbach resonance or at sufficiently large gas parameter, where observables can no longer be described by a_s alone.

From low-energy scattering to the Gross-Pitaevskii model

[A. Leggett, Oxford Univ. Press (2006)]
[L. P. Pitaevskii et al., Clarendon Press (2003)]
[L. Salasnich et al., *Phys. Rev. A* 65, 043614 (2002)]

Once the interaction is replaced by the contact coupling g , the condensate dynamics follows from the action

$$S = \int dt d^3\mathbf{r} \psi_0^* \left[i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2 - V_{\text{trap}} \right] \psi_0 - \frac{g}{2}|\psi_0|^4.$$

Its Euler-Lagrange equation is the three-dimensional Gross-Pitaevskii equation,

$$i\hbar\partial_t\psi_0 = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{trap}} + g|\psi_0|^2 \right] \psi_0.$$

Under strong transverse confinement, the Gaussian ansatz

$$\psi_0(\mathbf{r}, t) = f(z, t) \frac{1}{\sqrt{\pi} a_{\perp}} e^{-(x^2+y^2)/(2a_{\perp}^2)}$$

leads to the reduced equation

$$i\hbar\partial_t f = \left[-\frac{\hbar^2}{2m}\partial_z^2 + g_{1D}|f|^2 \right] f, \quad g_{1D} = \frac{g}{2\pi a_{\perp}^2}.$$

Here ψ_0 is the condensate order parameter, f the axial field, and $a_{\perp} = \sqrt{\hbar/(m\omega_{\perp})}$ the transverse oscillator length.

Projecting the two-body problem onto open and closed subspaces gives an effective interaction in the open channel,

$$\hat{V} = \hat{V}_1 + \hat{H}_{PQ}(E - \hat{H}_{QQ} + i\varepsilon)^{-1}\hat{H}_{QP},$$

so the closed-channel admixture enters as an energy-dependent Schur complement.

At low momentum this yields

$$\frac{4\pi\hbar^2}{m}a_s = \frac{4\pi\hbar^2}{m}a_{bg} + \frac{|\langle\psi_{\text{res}}|\hat{H}_{QP}|\psi_0\rangle|^2}{E - E_{\text{res}}},$$

and, after converting the detuning into magnetic-field units,

$$a_s(B) = a_{bg} \left(1 - \frac{\Delta B}{B - B_0} \right).$$

Here a_{bg} is the background scattering length, B_0 the resonance position, and ΔB the magnetic width of the resonance.

This is the mechanism that makes the low-energy coupling experimentally tunable, and it is the natural route toward regimes where effective-range terms cease to be negligible.

Keeping the first two terms of the low-momentum interaction expansion gives

$$V_{\text{eff}}(\mathbf{r}) = g\delta^{(3)}(\mathbf{r}) + \frac{g_2}{2} \left(\overleftarrow{\nabla}^2 \delta^{(3)}(\mathbf{r}) + \delta^{(3)}(\mathbf{r}) \overrightarrow{\nabla}^2 \right),$$

which leads to the modified mean-field equation

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{trap}} + g|\psi|^2 + \frac{g_2}{2}\nabla^2|\psi|^2 \right] \psi.$$

For lattice-soliton and collapse dynamics, three-body recombination adds an imaginary quintic term,

$$i\hbar\partial_t\psi = \dots - ig_5|\psi|^4\psi, \quad g_5 = \frac{\hbar N^2 L_3}{2}.$$

Here g_2 is the finite-range coupling, while g_5 and L_3 encode three-body loss.

The two corrections play different roles: g_2 deforms the conservative hydrodynamics, whereas g_5 regularizes high-density evolution by atom loss.

Propagation in a nonlinear dielectric starts from Maxwell equations with no free charges or currents,

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad \nabla \times \mathbf{H} = \partial_t \mathbf{D},$$

and constitutive relations

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{B} = \mu_0 \mathbf{H}.$$

Eliminating $\mathbf{H}$ gives the vector wave equation

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}.$$

Far from resonances one writes the polarization as

$$\mathbf{P} = \mathbf{P}_L + \mathbf{P}_{NL},$$

$$\mathbf{P}_L = \varepsilon_0 \int d\tau \chi^{(1)}(t - \tau) \mathbf{E}(\tau), \quad \mathbf{P}_{NL} = \varepsilon_0 \int d\tau_1 d\tau_2 d\tau_3 \chi^{(3)} \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3.$$

Here $\chi^{(1)}$ and $\chi^{(3)}$ are the linear and third-order susceptibilities, and the latter is responsible for Kerr nonlinearity.

Consider the D -dimensional NLSE with nonlinearity power σ :

$$i \frac{\partial}{\partial t} \psi + \nabla^2 \psi + |\psi|^{2\sigma} \psi = 0.$$

The energy functional (Hamiltonian) for a localized wave packet is

$$E[\psi] = \int d^D \mathbf{r} \left(|\nabla \psi|^2 - \frac{1}{\sigma + 1} |\psi|^{2\sigma+2} \right).$$

Under a norm-preserving scaling $\psi_\lambda(\mathbf{r}) = \lambda^{-D/2} \psi(\mathbf{r}/\lambda)$, the energy scales as

$$E(\lambda) = \frac{E_{kin}}{\lambda^2} - \frac{E_{int}}{\lambda^{D\sigma}},$$

where E_{kin} and E_{int} are positive constants.

The threshold for stability is determined by the balance of exponents:

- **Subcritical** ($D < 2/\sigma$): Dispersion (or kinetic energy) ($1/\lambda^2$) dominates at small λ .
- **Supercritical** ($D > 2/\sigma$): Attraction ($1/\lambda^{D\sigma}$) drives blow-up as $\lambda \rightarrow 0$.

The critical dimension is thus fixed at $D_c = 2/\sigma$. For cubic media ($\sigma = 1$), $D_c = 2$ represents the onset of self-focusing collapse.

The field is written as a carrier times a slowly varying envelope,

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{2} [E(\mathbf{r}, t)e^{-i\omega_0 t}] \hat{\mathbf{x}},$$

and for an instantaneous Kerr response one keeps

$$P_{NL}(\mathbf{r}, t) \simeq \varepsilon_0 \varepsilon_{NL} E, \quad \varepsilon_{NL} = \frac{3}{4} \chi^{(3)} |E|^2.$$

Projecting the Helmholtz equation onto a guided mode,

$$\tilde{E}(\mathbf{r}, \omega - \omega_0) = F(x, y) \tilde{A}(z, \omega - \omega_0) e^{i\beta_0 z},$$

and expanding the propagation constant as

$$\beta(\omega) \simeq \beta_0 + \beta_1(\omega - \omega_0) + \frac{\beta_2}{2}(\omega - \omega_0)^2,$$

one finds in the laboratory frame

$$\frac{\partial A}{\partial z} = -\beta_1 \frac{\partial A}{\partial t} - i \frac{\beta_2}{2} \frac{\partial^2 A}{\partial t^2} - \frac{\alpha}{2} A + i\gamma |A|^2 A.$$

Passing to the retarded time $T = t - \beta_1 z$ removes the drift and gives

For a cylindrical dielectric guide, separation of variables gives the longitudinal mode structure

$$\tilde{E}_z(\mathbf{r}, \omega) = A(\omega) F(\rho) e^{im\phi} e^{i\beta(\omega)z},$$

with the fundamental single-mode regime corresponding to the linearly polarized LP_{01} mode.

After the slowly-varying-envelope and narrow-band approximations, the axial envelope obeys

$$\frac{\partial A}{\partial z} = -\frac{\alpha}{2}A - i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + i\gamma|A|^2A, \quad \gamma = \frac{n_2\omega_0}{cA_{\text{eff}}}.$$

Here $F(\rho)$ is the transverse radial profile, $\beta(\omega)$ the propagation constant, α the loss coefficient, and A_{eff} the effective modal area.

This is the one-dimensional fiber NLSE used in the introduction. The backup point is that it comes from mode selection plus a controlled perturbative treatment of both dispersion and Kerr nonlinearity, rather than from an ad hoc scalar guess.

The nonlinear Schrödinger equation emerges generically for weakly nonlinear dispersive waves. Starting from

$$L(\partial_t, \nabla) u = G(u),$$

one first studies the linearized problem with plane-wave solution

$$u = \epsilon \psi e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad L(-i\omega, i\mathbf{k}) = 0.$$

Introducing slow variables

$$T = \epsilon t, \quad \mathbf{X} = \epsilon \mathbf{x},$$

and expanding around the carrier branch $\omega(\mathbf{k})$ yields

$$i(\partial_T + \mathbf{v}_g \cdot \nabla) \psi + \epsilon [\nabla \cdot (\mathcal{D} \nabla \psi) + \gamma |\psi|^2 \psi] = 0,$$

where $\mathbf{v}_g = \nabla_{\mathbf{k}} \omega$ is the group velocity and $\mathcal{D}$ is the half-Hessian of the dispersion relation. In the comoving frame $\zeta = \mathbf{X} - T \mathbf{v}_g$ and $\tau = \epsilon T$, the envelope satisfies

$$i \partial_\tau \psi + \nabla \cdot (\mathcal{D} \nabla \psi) + \gamma |\psi|^2 \psi = 0.$$

This derivation explains why the same equation recurs in optics, matter waves, hydrodynamics, and other weakly nonlinear wave systems.

For the generalized focusing NLSE

$$i\partial_t\psi + \nabla^2\psi + |\psi|^{2\sigma}\psi = 0,$$

two basic conserved quantities are

$$N = \int d^D\mathbf{r} |\psi|^2, \quad H = \int d^D\mathbf{r} \left[|\nabla\psi|^2 - \frac{1}{\sigma+1} |\psi|^{2\sigma+2} \right].$$

Its scaling symmetry is

$$\psi_\lambda(\mathbf{r}, t) = \lambda^{1/\sigma} \psi(\lambda\mathbf{r}, \lambda^2 t),$$

which identifies the critical combination

$$\sigma D = 2.$$

Therefore the dynamics is classified as

$$\sigma D < 2 \text{ subcritical}, \quad \sigma D = 2 \text{ critical}, \quad \sigma D > 2 \text{ supercritical}.$$

In the focusing case, collapse can occur only at and above the critical threshold; for the cubic nonlinearity $\sigma = 1$, this means the critical dimension is $D_c = 2$.

For radial interactions the Fourier transform collapses to one-dimensional kernels:

$$\tilde{f}(q) = 2 \int_0^\infty dx f(x) \cos(qx) \quad (D = 1),$$

$$\tilde{f}(q) = 2\pi \int_0^\infty dr r f(r) J_0(qr) \quad (D = 2),$$

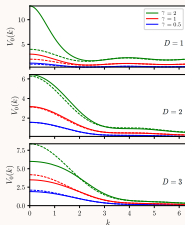
$$\tilde{f}(q) = 4\pi \int_0^\infty dr r^2 f(r) j_0(qr) \quad (D = 3).$$

The s-wave matrix element and the ordinary Fourier transform agree only at leading order:

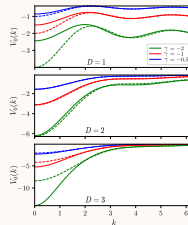
$$V_0(k) \sim g + g_2 k^2, \quad \tilde{V}(k) \sim \tilde{g} + \tilde{g}_2 k^2, \quad g = \tilde{g}, \quad g_2 = 2\tilde{g}_2.$$

Here J_0 and j_0 are the Bessel functions appearing in two and three dimensions, and tilded couplings come from the ordinary Fourier transform, whereas the untilded ones come from the s-wave matrix element.

That factor of two is exactly why the low-energy couplings entering local effective theories must be extracted from the partial-wave object instead of the naive Fourier coefficient alone.



Repulsive square well



Attractive square well

For

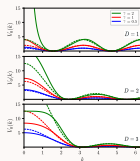
$$V(r) = \begin{cases} \gamma, & r \leq R, \\ 0, & r > R, \end{cases}$$

the exact phase shifts follow from matching at $r = R$, for instance

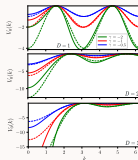
$$\delta_{0,3D}(k) = \arctan\left[\frac{k}{k_1} \tan(k_1 R)\right] - kR, \quad k_1 = \sqrt{k^2 - \frac{m\gamma}{\hbar^2}}.$$

Here R is the well radius, γ its depth or height, and k_1 the interior wave number.

At weak coupling the on-shell approximation reproduces the exact $V_0(k)$ to leading order in all dimensions, while at finite coupling it undershoots in $D = 1, 2$ and overshoots in $D = 3$.



Attractive shell



Repulsive shell

For the singular shell

$$V(r) = \gamma \delta^{(1)}(R - r),$$

continuity of ψ and the jump condition

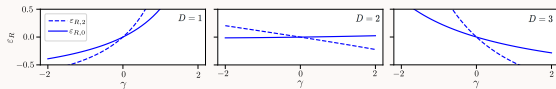
$$\psi'(R^+) - \psi'(R^-) = \frac{m}{\hbar^2} \gamma \psi(R)$$

replace the interior-momentum matching of the square well.

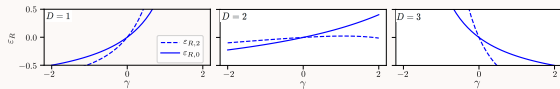
The exact phase shifts remain analytic; in three dimensions,

$$\delta_{0,3D}(k) = \arctan \left[\frac{k}{k \cot(kR) + \frac{m}{\hbar^2} \gamma} \right] - kR.$$

Here R is the shell radius and γ the shell strength.



Square well



Delta shell

These plots resolve separately the errors on

$$\tilde{g} = g, \quad \tilde{g}_2 = \frac{g_2}{2}.$$

Here $\tilde{g}$ and $\tilde{g}_2$ are the Fourier-space coefficients, while g and g_2 are the effective couplings extracted from the s-wave channel.

The practical message is narrow but important: the on-shell map is quantitatively reliable when the potential is weak and bound states are absent, and it remains qualitatively useful outside that regime because its bias is systematic and dimension dependent rather than erratic.

Finite-range equation of state in two dimensions

[S. R. Beane, *Eur. Phys. J. D* 72, 55 (2018)]
[C. Mora et al., *Phys. Rev. A* 67, 053615 (2003)]
[FL et al., *Phys. Rev. A* 107, 033325 (2023)]

The local low-momentum effective theory reads

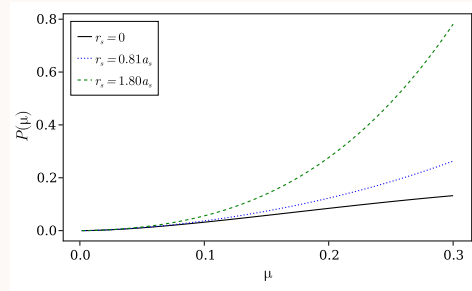
$$\mathcal{L} = \psi^* \left[i\hbar\partial_t + \frac{\hbar^2}{2m}\nabla^2 \right] \psi - \frac{\tilde{g}}{2}|\psi|^4 + \frac{\tilde{g}_2}{2}|\psi|^2\nabla^2|\psi|^2.$$

In two dimensions the couplings depend on the cutoff, while the one-loop pressure remains cutoff independent:

$$P(\mu) = \frac{m}{8\pi\hbar^2}\mu^2 \ln\left(\frac{4\hbar^2}{m\mu a_s^2 e^{2\gamma_E+1/2}}\right) + \frac{3m^2}{16\hbar^4}r_{\text{eff}}^2\mu^3.$$

Here μ is the chemical potential, P the pressure, a_s the scattering length, and r_{eff} the effective range.

The first term is the zero-range result, while the second isolates the leading finite-range correction.



With the Gaussian ansatz

$$\psi(\mathbf{r}) = \frac{1}{\pi^{3/4} \sigma \eta^{1/2}} \exp \left[-\frac{x^2 + y^2}{2\sigma^2} - \frac{z^2}{2\eta^2} \right],$$

the reduced energy becomes

$$E(\eta, \sigma) = \frac{1}{2} \left(\frac{1}{2\eta^2} + \frac{1}{\sigma^2} + \sigma^2 \right) + \frac{g}{2\sqrt{2\pi}} \frac{1}{\sigma^2 \eta}.$$

Adding the lattice contribution,

$$\Delta E_{\text{OL}}(\eta) = -V_0 e^{-k_L^2 \eta^2},$$

one gets a compact stability picture for single-site and multi-site localized packets.

Linearizing around a stationary point gives hybrid breathing frequencies

$$\Omega_{\pm} = \sqrt{\left(A + \frac{B}{2}\right) \pm \sqrt{\left(A + \frac{B}{2}\right)^2 - 4(AB - C^2)}}.$$

Here η and σ are the axial and transverse Gaussian widths, while A , B , and C are the Hessian coefficients of the variational energy at the stationary point.

Allowing the transverse Gaussian width to depend on z and t gives the variationally consistent pair

$$i\hbar\partial_t f = \left[-\frac{\hbar^2}{2m}\partial_z^2 + V + \frac{\hbar^2}{2m\sigma^2}(1 + (\partial_z\sigma)^2) + \frac{m\omega_\perp^2}{2}\sigma^2 + \frac{2\hbar^2 a_s(N-1)}{m\sigma^2}|f|^2 \right] f,$$

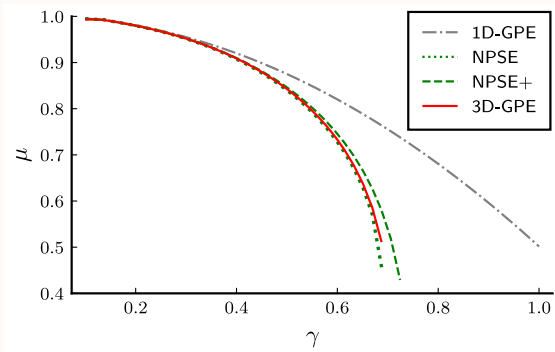
$$\sigma^4 - a_\perp^4 [1 + 2a_s(N-1)|f|^2] + a_\perp^4 [\sigma\partial_z^2\sigma - (\partial_z\sigma)^2 + \sigma\partial_z\sigma\partial_z\ln|f|^2] = 0.$$

Neglecting derivative terms recovers the standard NPSE closure

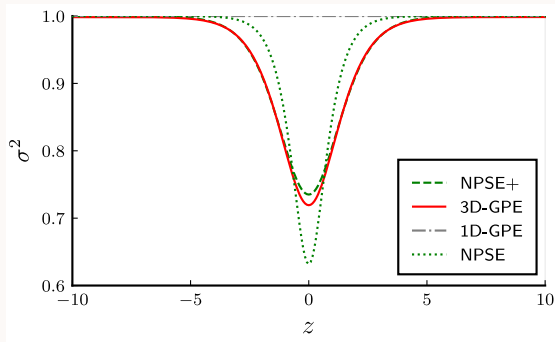
$$\sigma^2 = a_\perp^2 \sqrt{1 + 2a_s(N-1)|f|^2}.$$

Here $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$ is the transverse oscillator length and $\sigma(z, t)$ the local transverse width.

The point of the extended system is twofold: better numerics and an explicitly variational reduced model.

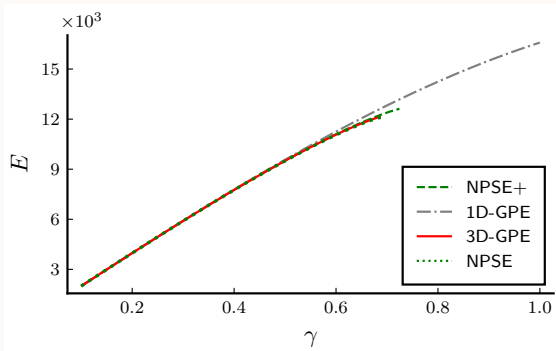


Chemical potential

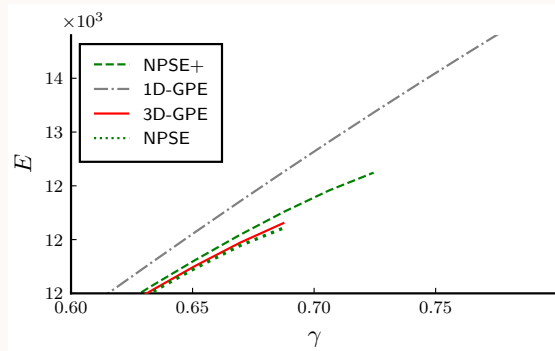


Transverse width

Close to γ_c , the 1D-GPE misses both the correct axial shape and the transverse squeezing. The NPSE reproduces the main trend, but its non-variational structure can drive the chemical potential below the three-dimensional value. The derivative corrections retained by NPSE+ repair that inconsistency and improve the transverse profile.



Full range

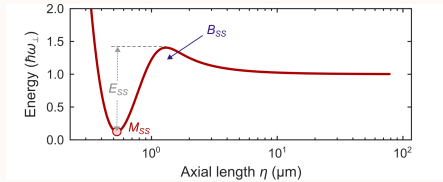
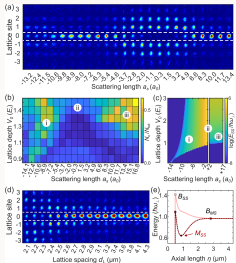


Near collapse

The energy comparison makes the variational point sharper than the density profiles alone. NPSE+ stays above the three-dimensional ground-state energy, as required by its construction, whereas the standard NPSE can slip below it in the strongly nonlinear regime. That is precisely where reduced theories are most tempting and most dangerous.

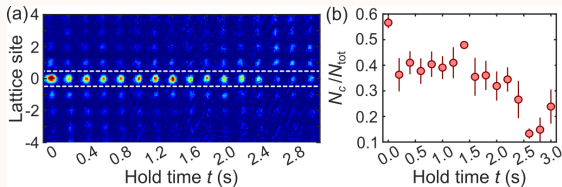
Single-site localization and repulsive self-trapping

[K. Winkler et al., *Nature* 441, 853–856 (2006)]
 [R. Cruickshank et al., *Phys. Rev. Lett.* 135, 263404 (2025)]
 [L. Salasnich et al., *Phys. Rev. A* 75, 033622 (2007)]

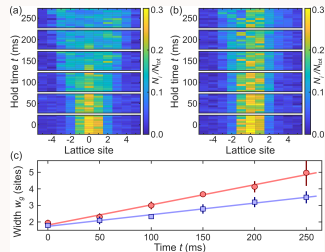


The single-site measurements resolve three regimes: an attractive localized state, a dispersive window around $a_s \simeq 0$, and a repulsively localized state at larger positive a_s .

The repulsive branch has a clear variational interpretation: the energy develops a local minimum M_{SS} protected by a barrier B_{SS} , so localization exist even when the bare interaction is defocusing.



Single-site time evolution



Multi-site occupations and widths

The single-site packet remains localized for nearly two seconds after the interaction quench, with the fast initial loss dominated by shedding and three-body recombination.

For multi-site packets, Gaussian fits give width velocities of roughly 12 sites/s for the dispersive case and 7 sites/s for the soliton-like case, showing that most of the broadening comes from profile flattening due to atom loss rather than from large mass transport.

GRIN fibers and higher-order Laguerre-Gauss reductions

[FL et al., *Nanophotonics* 14, 805–813 (2025)]

In a parabolic graded-index medium the envelope obeys a $(3 + 1)$ -dimensional NLSE,

$$i\partial_z\Phi = \left[-\frac{1}{2}(\partial_x^2 + \partial_y^2 + \partial_t^2) + W(x, y) + g|\Phi|^2 \right] \Phi.$$

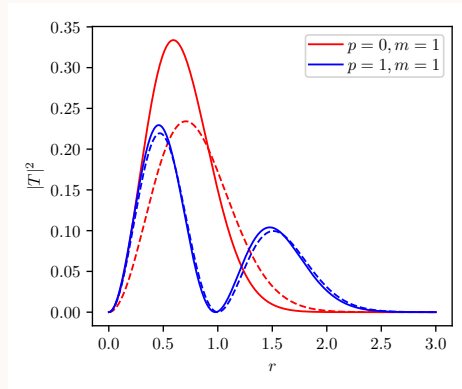
Using Laguerre-Gauss modes with radial index p and azimuthal index $S = |m|$ leads to a generalized NPSE,

$$i\partial_z A = -\frac{1}{2}\partial_t^2 A + \xi_{pS} \left(\frac{1 + \frac{3}{2}g_{pS}|A|^2}{\sqrt{1 + g_{pS}|A|^2}} \right) A,$$

with $\xi_{pS} = S + 2p + 1$.

Here A is the reduced axial envelope, while p and S label the radial and azimuthal Laguerre-Gauss mode numbers.

This reduction keeps arbitrary axial pulse shapes while encoding the transverse mode dependence in a small set



Expanding the nonpolynomial potential produces the cubic and cubic–quintic reductions,

$$E_3 = \frac{2\sqrt{2}}{\gamma_{pS}} \sqrt{1 + \kappa_{pS}}, \quad E_{3-5} = \frac{2}{\gamma_{pS}} \arctan(\sqrt{2}\sqrt{1 + \kappa_{pS}}).$$

In the full NPSE the stationary branch obeys

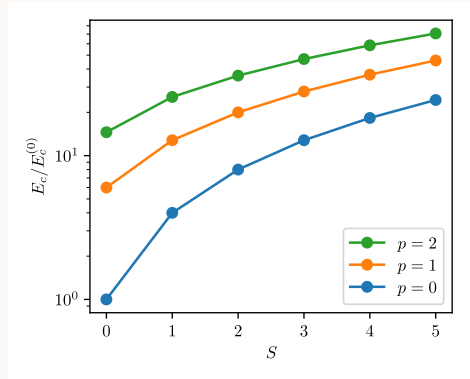
$$E = \frac{2\sqrt{2}}{3\gamma_{pS}} (2\kappa_{pS} + 1) \sqrt{1 - \kappa_{pS}},$$

so the Vakhitov–Kolokolov condition gives

$$E_c = \frac{4}{3\gamma_{pS}}.$$

Here κ_{pS} is the stationary variational parameter and γ_{pS} the effective nonlinear coupling for mode (p, S) .

The cubic model never predicts collapse, the cubic–quintic



Josephson frequency correction in two dimensions

[A. Bardin et al., *New J. Phys.* 26, 013021 (2024)]

Using the renormalized two-dimensional coupling, the oscillation frequency becomes

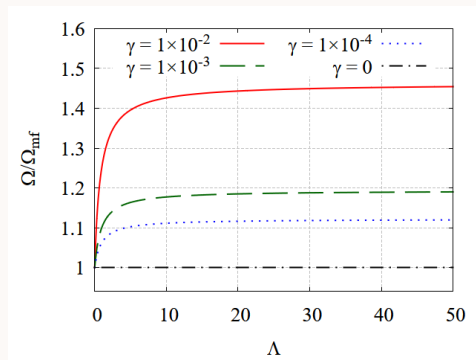
$$\Omega = \frac{1}{\hbar} \sqrt{J^2 + J g_r n \left(1 - \frac{3}{2 \ln(Cn)} + \frac{1}{\ln^2(Cn)} \right)}.$$

In terms of the gas parameter $\gamma = a_s^2 n$,

$$\frac{\Omega}{\Omega_{\text{mf}}} = \sqrt{\frac{1 + \Lambda \left(1 - \frac{3}{2 \ln(C^* \gamma)} + \frac{1}{\ln^2(C^* \gamma)} \right)}{1 + \Lambda}}.$$

Here g_r is the renormalized two-dimensional coupling, $\gamma = a_s^2 n$ the gas parameter, and C^* the dimensionless constant that absorbs the scattering-scale convention.

The correction is positive, stronger at larger γ , and parametrically larger than in three dimensions at



Away from the small-imbalance limit, the interaction term acquires explicit z dependence,

$$U_k = U_r \left[1 + \frac{\ln(1 \pm z)}{\ln(Cn)} \right]^{-1},$$

so the critical strength becomes

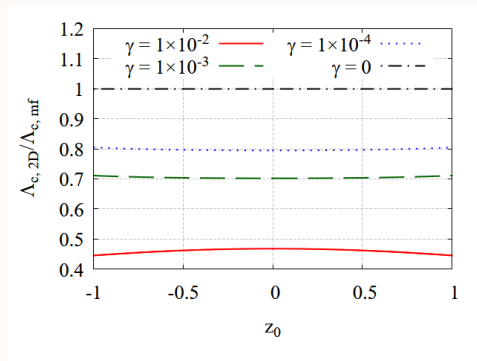
$$\Lambda_r > \Lambda_{c,2D}.$$

Relative to mean field,

$$\frac{\Lambda_{c,2D}}{\Lambda_{c,mf}} < 1,$$

with the reduction growing as the gas parameter increases.

Here U_k is the imbalance-dependent interaction coefficient, U_r its reference renormalized value, and Λ_r the corresponding dimensionless interaction strength



Superconducting system, an approach using an only-phase action

[V. N. Popov, Reidel, Dordrecht (1983)]
[D. T. Son et al., *Ann. Phys.* 321, 197 (2006)]
[L. Salasnich et al., *J. Phys. A: Math. Th.* 57, 355302 (2024)]

Starting from the density-phase action

$$S[n, \theta] = \int dt d^D \mathbf{r} \left[-\mathcal{E}_0(n) - n\hbar\partial_t\theta - n\frac{\hbar^2}{2m}|\nabla\theta|^2 + \mu n \right],$$

with $\mathcal{E}_0(n)$ the internal energy density ($= gn^2/2$ for weakly interacting bosons) the saddle-point density satisfies

$$\mu - \hbar\partial_t\theta - \frac{\hbar^2}{2m}|\nabla\theta|^2 = \frac{\partial\mathcal{E}_0}{\partial n}(n_s).$$

Eliminating n by Legendre transform gives the exact phase-only Lagrangian

$$\tilde{\mathcal{L}} = P\left(\mu - \hbar\partial_t\theta - \frac{\hbar^2}{2m}|\nabla\theta|^2\right).$$

Here $\mathcal{E}_0(n)$ is the bulk energy density, n_s the saddle-point density, and $P(\mu)$ the pressure as Legendre transform of the energy functional.

Expanding around equilibrium recovers the usual hydrodynamic action, but the key conceptual point is that the pressure functional itself is the generator of the low-energy phase theory.

For a complex Klein–Gordon field one finds, after removing the rest-energy oscillation and neglecting density-gradient terms,

$$\mathcal{L}_R = n \frac{\hbar^2}{2mc^2} (\partial_t \theta)^2 - n \hbar \partial_t \theta - n \frac{\hbar^2}{2m} |\nabla \theta|^2 - \mathcal{E}_0(n) + \mu n.$$

The corresponding only-phase form is

$$\tilde{\mathcal{L}}_R = P \left(\mu + \frac{\hbar^2}{2mc^2} (\partial_t \theta)^2 - \hbar \partial_t \theta - \frac{\hbar^2}{2m} |\nabla \theta|^2 \right).$$

Here $\mathcal{L}_R$ is the relativistic density-phase Lagrangian and c_s is the sound velocity entering the low-energy expansion.

Its expansion contains the nonrelativistic action plus an additional temporal-stiffness term proportional to $(1 + c^2/c_s^2)(\partial_t \theta)^2$.

Only-phase action coupled to the electromagnetic field

[A. M. J. Schakel, *Mod. Phys. Lett. B* 4, 927 (1990)]
[L. Salasnich et al., *J. Phys. A: Math. Th.* 57, 355302 (2024)]
[J. E. Hirsch, *Phys. Rev. B* 69, 214515 (2004)]

Minimal coupling amounts to

$$\frac{\partial \theta}{\partial t} \rightarrow \frac{\partial \theta}{\partial t} + \frac{q}{\hbar} \Phi, \quad \nabla \theta \rightarrow \nabla \theta - \frac{q}{\hbar} \mathbf{A},$$

so that the total Lagrangian density becomes

$$\mathcal{L}_{\text{tot}} = P(\mu_{\text{eff}}^{\text{em}}) + \frac{\epsilon_0}{2} |\mathbf{E}|^2 - \frac{1}{2\mu_0} |\mathbf{B}|^2 + q\bar{n}_{\text{bg}} \Phi,$$

with

$$\mu_{\text{eff}}^{\text{em}} = \mu - \hbar \frac{\partial \theta}{\partial t} - q\Phi - \frac{1}{2m} (\hbar \nabla \theta - q\mathbf{A})^2.$$

The Euler-Lagrange equations with respect to Φ and $\mathbf{A}$ are Maxwell equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial}{\partial t} \mathbf{E},$$

with constitutive relations

$$\rho = qn - q\bar{n}_{\text{bg}} - \epsilon_0 \frac{q^2 n \mu_0}{m} \frac{c^2}{c_s^2} \Phi, \quad \mathbf{j} = qn \mathbf{v}_s - \frac{q^2 n}{m} \mathbf{A}.$$

In words: the vector potential produces the current, while the scalar potential also enters the charge density through the Popov correction. That additional term is what changes the electric screening scale.