A couple of algorithms for numerical optics: Gerchberg–Saxton phase retrieval and “Fast” Hankel transforms

Gerchberg–Saxton Phase Retrieval

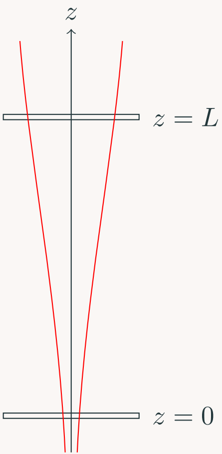
Phase retrieval problem statement

- Phase retrieval is the task to **reconstruct complex fields** from **intensity-only** data.

We seek a complex scalar field (e.g. proportional to one polarization of the electric field) $u(\mathbf{r}_\perp, z = 0)$ given intensity constraints in two planes:

$$I(\mathbf{r}_\perp, z = 0) = |u(\mathbf{r}_\perp, z = 0)|^2,$$
$$I(\mathbf{r}_\perp, z = L) = |u(\mathbf{r}_\perp, z = L)|^2.$$

The field $u(\mathbf{r}_\perp, z = L)$ is obtained by propagating $u(\mathbf{r}_\perp, z = 0)$ between planes within a given model for scalar diffraction.



Propagation operator

The general relationship between complex fields is written in terms of a unitary operator:

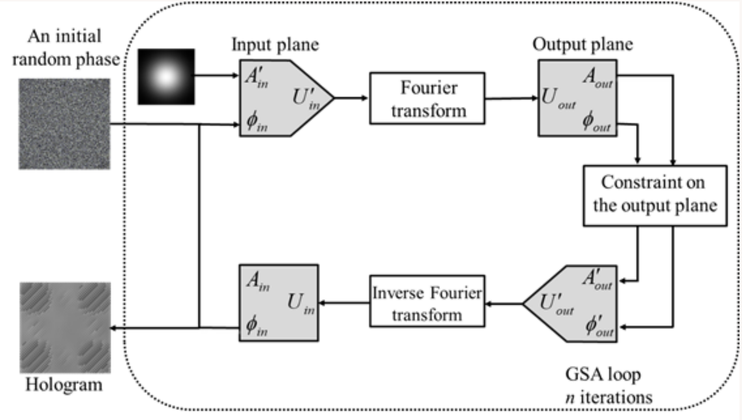
$$u(\mathbf{r}_\perp, z = L) = \mathsf{U}_L u(\mathbf{r}_\perp, z = 0).$$

A quite general one is the **angular spectrum method**, which leads to the paraxial propagation and to Fraunhofer diffraction in the far field case.

The phase retrieval setting resembles a **boundary value problem** rather than an **initial value problem**.
However, I wasn't able to find simple mathematical results about the **uniqueness** of the solution (modulo a global phase).

Does **uniqueness** naturally requires **superresolution**? It may be possible within ASM due to the evanescent field component.

Gerchberg–Saxton loop



[1]

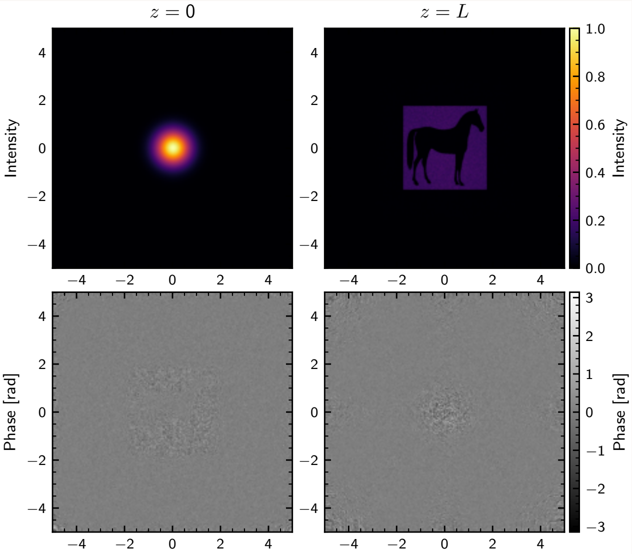
[1] R. W. Gerchberg and W. O. Saxton, *Optik* **35**, 237 (1972).

Algorithm sketch (ASM)

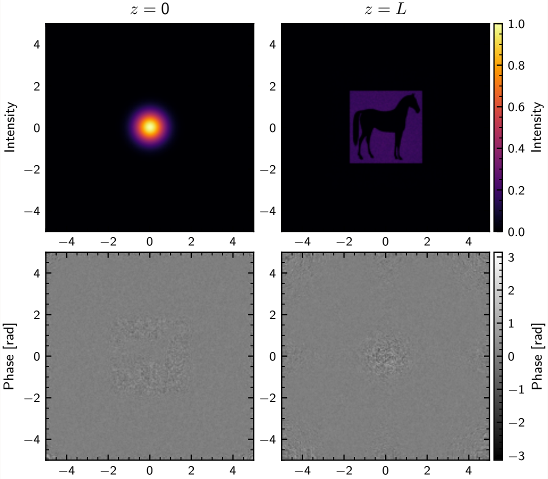
1. Initialize $u^{(0)}(\mathbf{r}_\perp, z = 0)$ with measured magnitude $\sqrt{I(\mathbf{r}_\perp, z = 0)}$ and random phase.
2. Propagate to plane $z = L$: $u^{(n)}(\mathbf{r}_\perp, z = L) = \mathsf{U}_L u^{(n)}(\mathbf{r}_\perp, z = 0)$.
3. Enforce magnitude at $z = L$: $\widetilde{u}^{(n)}(\mathbf{r}_\perp, z = L) = \sqrt{I(\mathbf{r}_\perp, z = L)}\, e^{i\phi^{(n)}(\mathbf{r}_\perp, z=L)}$.
4. Back-propagate: $\widetilde{u}^{(n)}(\mathbf{r}_\perp, z = 0) = \mathsf{U}_{-L}\widetilde{u}^{(n)}(\mathbf{r}_\perp, z = L)$.
5. Enforce object magnitude: $u^{(n+1)}(\mathbf{r}_\perp, z = 0) = \sqrt{I(\mathbf{r}_\perp, z = 0)}\, e^{i\phi^{(n)}(\mathbf{r}_\perp, z=0)}$.

Numerical experiments

- Grid: 2048×2048.
- 20 GS iterations.
- Propagation over $L = 10^{-2} z_R$ ($z_R = \frac{\pi w_0^2}{\lambda_0}$).
- Transverse length normalized to $w_0 = 10^3 \lambda_0$ initial Gaussian width.
- Runs in ~ 10s on in i7-12700 16GB laptop.

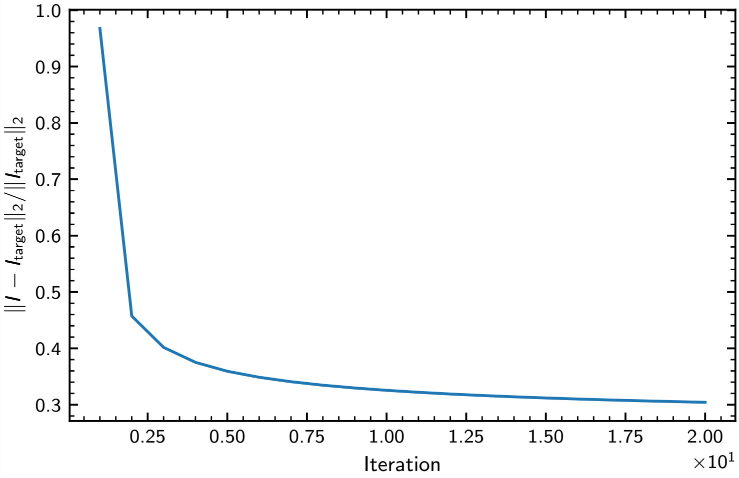


How does the diffraction look like?



horse_shape_propagation.gif

Speed of convergence

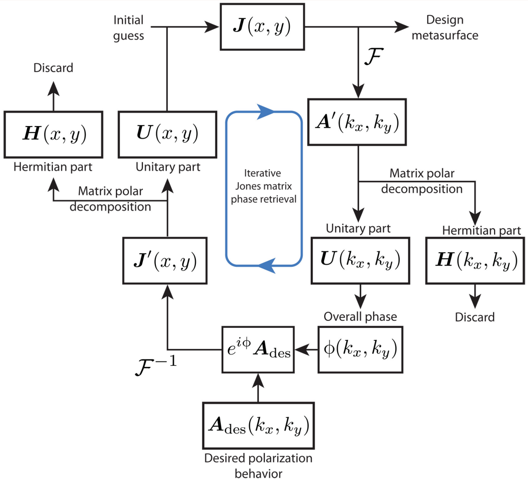


Applications and scope

- Pulse shaping: retrieve phase modulation from measured intensity constraints.
- Coherent diffraction imaging and ptychography.
- Wavefront sensing and optical testing.
- Holography and beam shaping with SLMs.

This is only a starting point; the literature contains many alternative algorithms and variants tailored to specific constraints.

GS for Jones matrix holography



[2]

[2] N. A. Rubin *et al.*, *Sci. Adv.* **7**, eabg7488 (2021).

“Fast” Hankel Transforms

Helmholtz equation and diffraction

- Diffractive propagation is naturally formulated in the Fourier wavevector domain.
- Radially symmetric systems with radially symmetric input are very common and may be studied in a reduced setting.

The scalar field envelope E satisfies

$$(\nabla^2 + \kappa^2)E = 0,$$

and its Fourier components obey the dispersion relation

$$k_x^2 + k_y^2 + k_z^2 = \kappa^2.$$

where $\kappa = n\omega/c$ is the wavenumber in the linear, isotropic and homogeneous medium.

Paraxial propagation (spectral)

If A is the complex envelope in z, in the paraxial approximation the envelope spectrum evolves as

$$\widehat{A}(k_x, k_y, z) = \widehat{A}(k_x, k_y, 0) \exp\left[-i\frac{k_x^2 + k_y^2}{2\kappa} z\right].$$

Define the propagator

$$\widehat{\mathsf{U}}(k_x, k_y, z) = \exp\left[-i\frac{k_x^2 + k_y^2}{2\kappa} z\right],$$

so that

$$A(\mathbf{r}_\perp, z_n) = \mathsf{F}_\perp^{-1}\left[\widehat{\mathsf{U}}(k_x, k_y, z_n)\, \mathsf{F}_\perp[A(\mathbf{r}_\perp, 0)]\right].$$

FFT-based evaluation on an $N \times N$ grid scales as $\mathcal{O}(N^2 \log N)$, while direct quadrature scales as $\mathcal{O}(N^4)$. We do have exact solutions, for example for Gaussian and Bessel beams.

Circular symmetry and Hankel transform

When $A(\mathbf{r}_\perp, z) = A(r, z)$, the transverse transform becomes

$$\begin{aligned}\mathsf{F}[A(\mathbf{r}_\perp, z)](\mathbf{k}_\perp, z) &= \int d^2\mathbf{r}_\perp \; rA(r,z)\exp\left[-i\mathbf{k}_\perp \cdot \mathbf{r}_\perp\right] \\ &= 2\pi \int_0^\infty dr \; rA(r,z)J_0(kr) \\ &= \mathsf{H}[A(r,z)](k,z).\end{aligned}$$

Can we evaluate this Fourier–Bessel (Hankel) integral efficiently, in a way analogous to the FFT? Naive quadrature scales as $\mathcal{O}(N^2)$, while with a fast method we may aim at $\mathcal{O}(N \log N)$.

Here is a fast method!

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Quasi fast Hankel transform

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Received March 25, 1977

We outline here a new algorithm for evaluating Hankel (Fourier–Bessel) transforms numerically with enhanced speed, accuracy, and efficiency. A nonlinear change of variables is used to convert the one-sided Hankel transform integral into a two-sided cross-correlation integral. This correlation integral is then evaluated on a discrete sampled basis using fast Fourier transforms. The new algorithm offers advantages in speed and substantial advantages in storage requirements over conventional methods for evaluating Hankel transforms with large numbers of points. [3]

[3] A. E. Siegman, *Opt. Lett.* 1, 13 (1977).

Logarithmic coordinate transformation

The standard Hankel transform of order ℓ is given by

$$g(\rho) = 2\pi \int_0^\infty r f(r) J_\ell(2\pi\rho r)\, dr \tag{1}$$

with a symmetric reverse transform integral from $g(\rho)$ to $f(r)$. Using the change of variables $r = r_0 e^{\alpha x}$, $\rho = \rho_0 e^{\alpha y}$, where r_0, ρ_0, and α are initially arbitrary, converts the one-sided Hankel transform integral [Eq. (1)] into the two-sided cross-correlation integral

$$\widehat{g}(y) = \int_{-\infty}^\infty \widehat{f}(x)\, \widehat{J}(x+y)\, dx. \tag{2}$$

this can be evaluated with FFT methods.

The problem with accuracy: wise solution of the sampling points

Magni *et al.* Vol. 9, No. 11/November 1992/J. Opt. Soc. Am. A 2031

High-accuracy fast Hankel transform for optical beam propagation

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Received March 3, 1992; accepted May 6, 1992

We describe a new method for the numerical calculation of the zero-order Hankel (Fourier–Bessel) transform that has a high computational efficiency and an accuracy that can be 2 orders of magnitude greater than that of the standard quasi-fast Hankel procedure. The new method offers particular advantages in calculating optical beam propagation and resonator modes at high Fresnel numbers. [4]

[4] V. Magni *et al.*, *JOSA A* **9**, 2031 (1992).

Saving the numerical Parseval relation

March 15, 1998 / Vol. 23, No. 6 / OPTICS LETTERS **409**

Quasi-discrete Hankel transform

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Received November 24, 1997

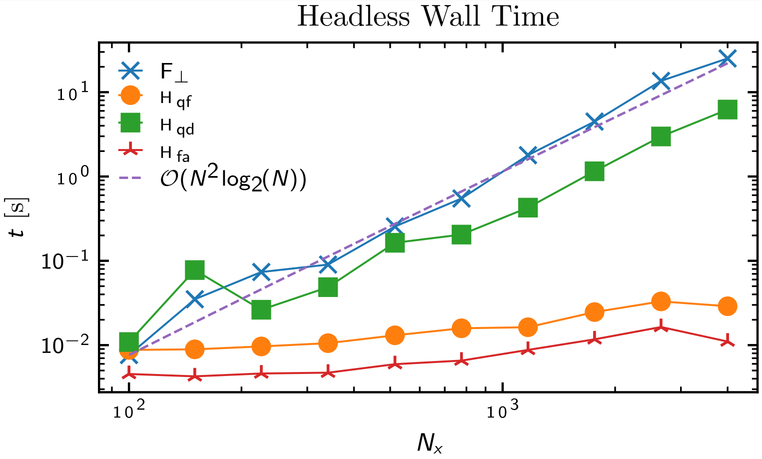
A quasi-discrete Hankel transform (QDHT) is presented as a new and efficient framework for numerical evaluation of the zero-order Hankel transform. A discrete form of Parseval's theorem is obtained for the first time to the authors' knowledge, and the transform matrix is discussed. It is shown that the S factor, defined as the products of a truncated radius, is critical to building the QDHT.

OCIS codes: 000.5360, 070.2590.

[5]

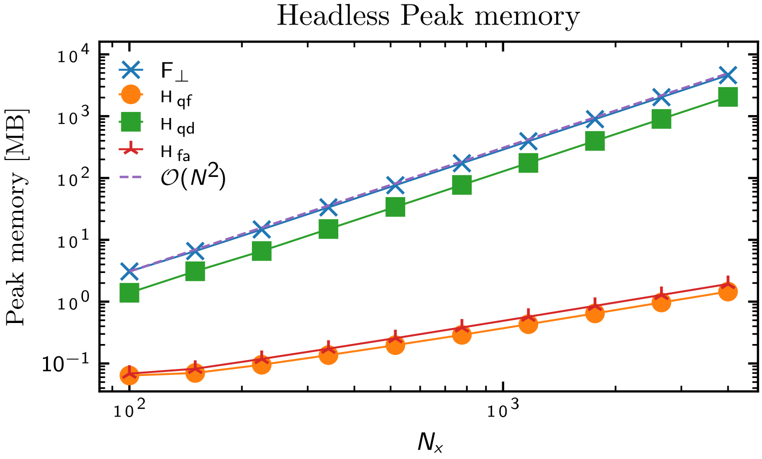
[5] L. Yu *et al.*, *Opt. Lett.* **23**, 409 (1998).

Performance (time)



2D-FFT-based propagation on an $N \times N$ grid scales as $\mathcal{O}(N^2 \log N)$, while direct evaluation scales as $\mathcal{O}(N^4)$.

Performance (memory)



Design flow chart

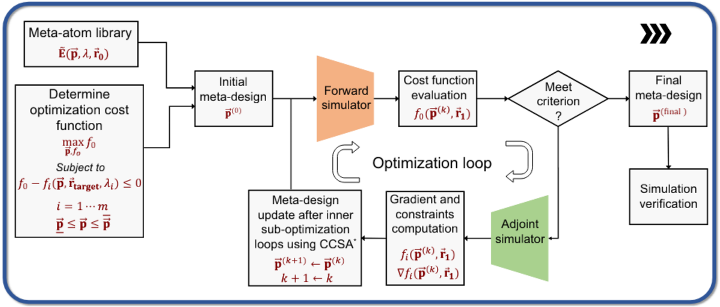


Fig. 2 Meta-optics inverse-design flow chart. With prior knowledge of the meta-atom library and optimization problem, we start with a random metasurface design and then update the design through optimization loops that consists of a forward simulator and an adjoint-based optimization engine. Once the criterion is met, we terminate the design loop and validate the design in simulation. Note: CCSA is short for conservative convex separable approximation. [6]

[6] Z. Li *et al.*, *Nat. Commun.* **13**, 2409 (2022).

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Thanks for the attention!

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Thanks for the attention!

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