Variational approach to multimode nonlinear optical fibers

Francesco Lorenzi [1, 2]   Luca Salasnich [1, 2, 3, 4]

[1]Dipartimento di Fisica e Astronomia “Galileo Galilei”, Università di Padova (Italy)
[2]Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Padova (Italy)
[3]Padua Quantum Technology Research Center, Università di Padova (Italy)
[4]Istituto Nazionale di Ottica (INO-CNR) (Italy)

Nonlinear dynamics in graded-index fibers

We consider light propagation in a **parabolic graded-index** (GRIN) multimode optical fiber, in the nonlinear regime.

1. Dynamics is effectively described using the $(3+1)$D nonlinear Schrödinger equation with **Kerr** nonlinearity and dispersion up to the second order.
2. A **variational ansatz** is formulated based on Laguerre–Gauss modes with variable transverse width.
3. By dimensional reduction, we obtain **effective one-dimensional equations** for the axial field and width, keeping the axial field as a free function. These are of the form of generalized Nonpolynomial Schrödinger equation (NPSE).

Consider the Helmholtz equation

$$\boxed{\left[\nabla^2 + \beta^2(\omega)\right] \tilde{\mathbf{E}}(\mathbf{r},\omega) = \mathbf{0}},$$

with the choice

$$\beta^2(\omega) = \delta\, \omega^2 - 2W(x,y) - 2g \int \tilde{\mathbf{E}}^*(\mathbf{r},\omega')\cdot\tilde{\mathbf{E}}(\mathbf{r},\omega-\omega')\, d\omega' ,$$

$$W(x,y) \propto (x^2+y^2)/\ell_\perp^2 .$$

Effective nonlinear Schrödinger equation for the light field

We identify a slowly-varying field Φ,

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \frac{1}{2}\tilde{\Phi}(\mathbf{r},\omega)\, e^{i\beta_0 z}\, \mathbf{u} + \text{c.c.},$$

and rewrite the Helmholtz equation into the NLSE

$$i\beta_0\partial_z\Phi(\mathbf{r},t) = \left[-\frac{1}{2}\nabla_\perp^2 + \frac{\delta}{2}\partial_t^2 + \frac{1}{2}\beta_0^2 + W(x,y) + g|\Phi(\mathbf{r},t)|^2 \right]\Phi(\mathbf{r},t)$$

Comparing this equation with the NLSE of quantum mechanics (Gross-Pitaevskii equation), there is an **exchange of the axial coordinate z with the time coordinate t**.
With $\delta<0$ – the **anomalous dispersion** case – we obtain the following adimensional NLSE

$$\boxed{i\partial_z\Phi(x,y,z,t) = \left[-\frac{1}{2}(\partial_x^2+\partial_y^2+\partial_t^2) + W(x,y) + g|\Phi(x,y,z,t)|^2 \right]\Phi(x,y,z,t)} \qquad (1)$$

A variational approach to dimensional reduction

The previous NLSE can be interpreted as the Euler-Lagrange equation obtained by extremizing the action functional

$$S[\Phi] = \int L\, dt ,$$

with Lagrangian

$$L = \int \mathscr{L}\, dx\, dy\, dz ,$$

and Lagrangian density

$$\boxed{\mathscr{L} = \tfrac{i}{2}(\Phi^*\partial_z\Phi - \Phi\partial_z\Phi^*) - \frac{1}{2}(|\partial_x\Phi|^2 + |\partial_y\Phi|^2 + |\partial_t\Phi|^2) - W(x,y)|\Phi|^2 - \frac{g}{2}|\Phi|^4},$$

So, there is a strong **analogy with the Gross-Pitaevskii theory** that describes the mean-field properties of a Bose-Einstein condensate. In particular, when considering the case of **anomalous dispersion**, the Lagrangian density corresponds to the one of an **attractive condensate**.

In absence of Kerr nonlinearity, the mode structure of the propagation equation is described in terms of **Laguerre-Gauss modes** indexed by two integer numbers $n = 0, 1, 2, \ldots$ and $m = -n, \ldots, n$, with $S = |m|$.

$$\boxed{\Phi_{nm}(r,\theta,z,t) = A_{nS}(z,t)\, T_{nS}(r;\sigma_{nS}(z,t))\, e^{im\theta}}.$$

We can assume $\sigma_{nS}(z,t)$ as a **variational parameter**. The transverse functions are

$$T_{nS}(r,\sigma_{nS}(z,t)) = \sqrt{\frac{p!}{\pi\sigma_{nS}^2(z,t)(p+S)!}} \times \left(\frac{r}{\sigma_{nS}(z,t)}\right)^S \exp\left[-\frac{r^2}{2\sigma_{nS}^2(z,t)}\right] L_p^S\left(\frac{r^2}{\sigma_{nS}^2(z,t)}\right),$$

where n is the **principal integer number**, m is the **angular integer number**, and $p = (n-S)/2$ is the **radial number**.
Special cases are the *ansätze* of the form

$$T_{SS}(r,\sigma_{SS}(z,t)) = \frac{r^S}{\sqrt{\pi S!}\, \sigma_{SS}^{S+1}(z,t)} \exp\left[-\frac{r^2}{2\sigma_{SS}^2(z,t)}\right]$$

which are the Laguerre-Gauss modes under the condition of a **nodeless radial profile**. This is reminiscent of the treatment of vortices in Bose-Einstein condensates, where the condensate density near the origin follows a power law with an exponent equal to the vorticity. It is a class of Laguerre-Gauss states in which $n = S$, or $p = 0$, indicating a **single maximum** of the transverse function. This state will correspond to a **simple vortical state** with vorticity m.

Quasi one-dimensional effective equations

Neglecting the derivatives of the transverse width σ, the effective Lagrangian is obtained as

$$\mathscr{L}^{(p)} = iA^*\partial_z A - \frac{1}{2}|\partial_t A|^2 - \frac{1}{2}\xi_{pS}\left(\left(\frac{1}{\sigma^2}+\sigma^2\right)|A|^2 + g_{pS}\frac{|A|^4}{\sigma^2}\right) .$$

with $\xi_{pS} = S + 2p + 1$, and the **effective interaction strength**

$$g_{pS} = \frac{g}{2\pi}\frac{1}{\xi_{pS}2^{4p+2S}}\sum_{q=0}^{p}\frac{(2q)![(2p-2q)!]^2(2S+2q)!}{(q!)^2[(p-q)!]^4[(S+q)!]^2}.$$

The corresponding Euler-Lagrange equations consist of a (1+1)-dimensional PDE for the **axial field** A

$$i\partial_z A = -\frac{1}{2}\partial_t^2 A + \xi_{pS}\left(\frac{1+\sigma^4+2g_{pS}|A|^2}{2\sigma^2}\right)A,$$

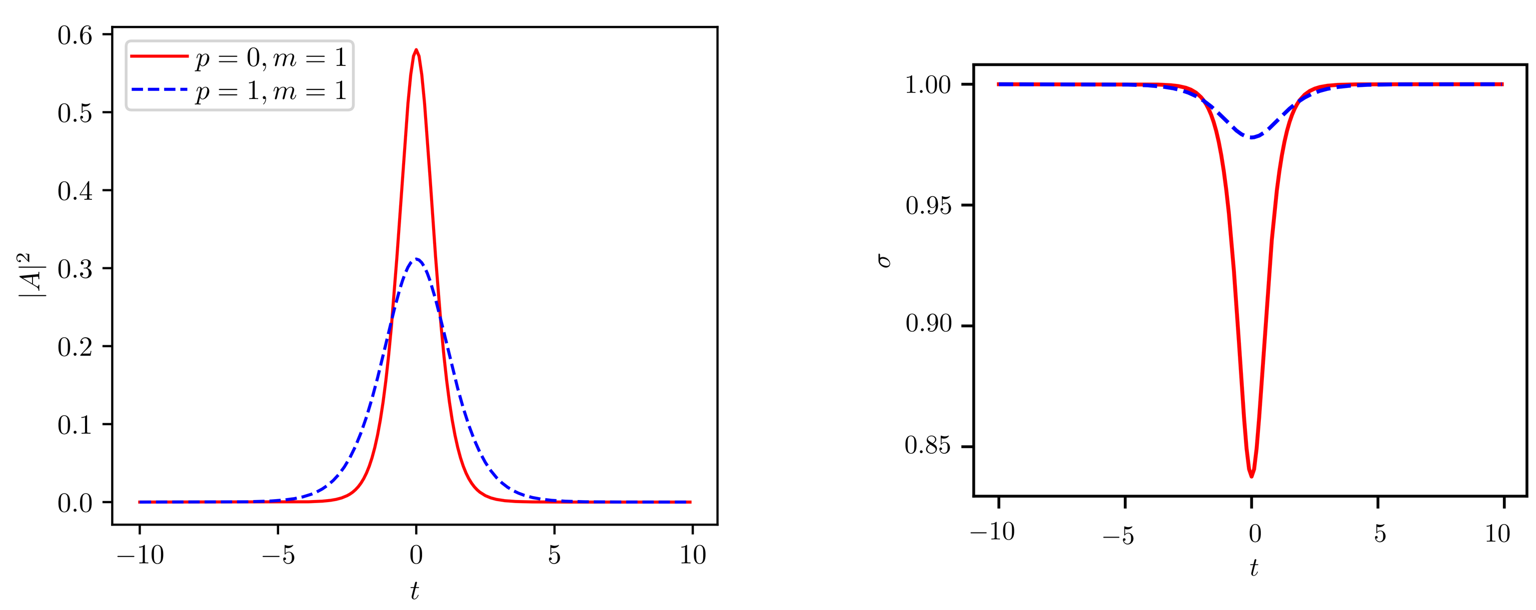
and an algebraic equation for the **variational width** σ,

$$\sigma^4 = 1 + g_{pS}|A|^2 .$$

Since the Euler-Lagrange equation for the function σ is indeed algebraic, we can substitute the value of σ back into the axial field equation, to obtain the **NPSE for the Laguerre-Gauss modes**,

$$\boxed{i\partial_z A = -\frac{1}{2}\partial_t^2 A + \xi_{pS}\left(\frac{1+(3/2)g_{pS}|A|^2}{\sqrt{1+g_{pS}|A|^2}}\right)A}.$$

Soliton-like solutions



(a) Pulse intensity $|A|^2$ of the solitonic solutions.   (b) Transverse width σ.

Within the **stationary ansatz** $A(z,t) = a(t)\exp[-i\kappa z]$, and with normalized interaction parameter $\gamma_{pS} = |g_{pS}|$, and a rescaled propagation variable $\kappa_{pS} = \kappa/\xi_{pS}$, we obtain the stationary equation

$$\kappa a = -\frac{1}{2}\partial_t^2 a + \xi_{pS}\left(\frac{1+(3/2)g_{pS}|a|^2}{\sqrt{1+g_{pS}|a|^2}}\right)a ,$$

and determine the temporal shape of the soliton via quadrature. Indeed, by imposing that the soliton amplitude vanishes for $t \to \pm\infty$, we obtain the **implicit relation**

$$\boxed{t = \frac{1}{\sqrt{2\,\xi_{pS}}}\left[\frac{1}{\sqrt{1-\kappa_{pS}}}\arctan\left(\sqrt{\frac{\sqrt{1-\gamma_{pS}a^2}-\kappa_{pS}}{1-\kappa_{pS}}}\right) - \frac{1}{\sqrt{1+\kappa_{pS}}}\operatorname{arctanh}\left(\sqrt{\frac{\sqrt{1-\gamma_{pS}a^2}-\kappa_{pS}}{1+\kappa_{pS}}}\right)\right]}.$$

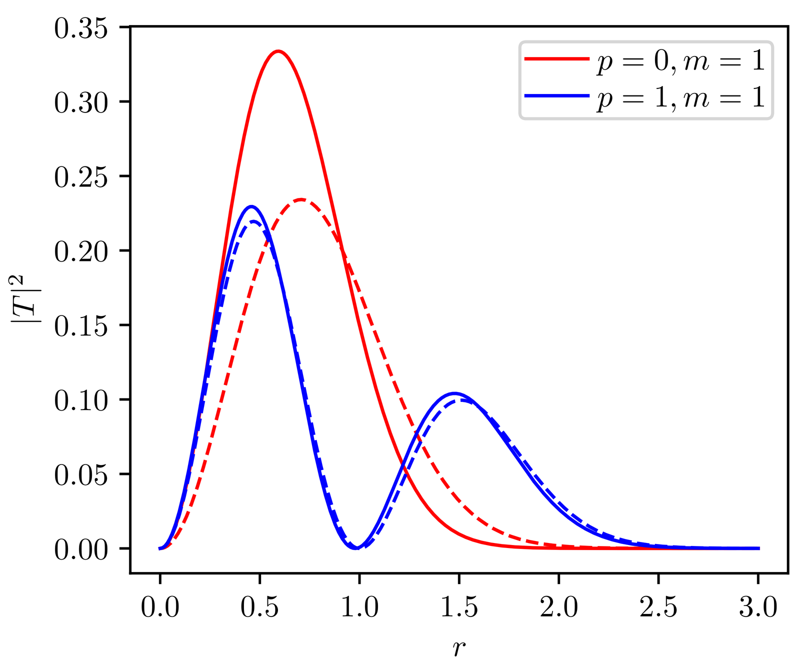


Figure 2. Radial profile for two solitonic solutions of the stationary NPSE Eq. (1), corresponding to the $m = 1$ states with $p = 0$ (red lines) and $p = 1$ (blue lines). The solid line is the radial shape at the time corresponding to the peak of the soliton, and the dashed line corresponds to the linear case, which approximates the low-energy regime. r in units of $\ell_\perp$.

Stability of the solitons and Cubic-quintic approach

We impose the normalization condition for the pulse normalized energy $E = \int_{-\infty}^{\infty} dt|A(z,t)|^2$ therefore obtaining a relation between the **propagation variable** and the **nonlinearity parameter**

$$E = \tfrac{2\sqrt{2}}{3\gamma_{pS}}(2\kappa_{pS}+1)\sqrt{1-\kappa_{pS}}.$$

This relation has solutions for every $E < E_c$ that is the threshold energy for the collapse instability of the optical field. Moreover, this point is the point of union of the stable and unstable branches as marked by the **Vakhitov-Kolokolov (VK) criterion**. The VK criterion is a **necessary condition** for the stability of the solitons, that is, the solitons are stable only if $\partial E/\partial\kappa_{pS} > 0$, yielding

$$\boxed{E_c = \frac{4}{3\gamma_{pS}}}.$$

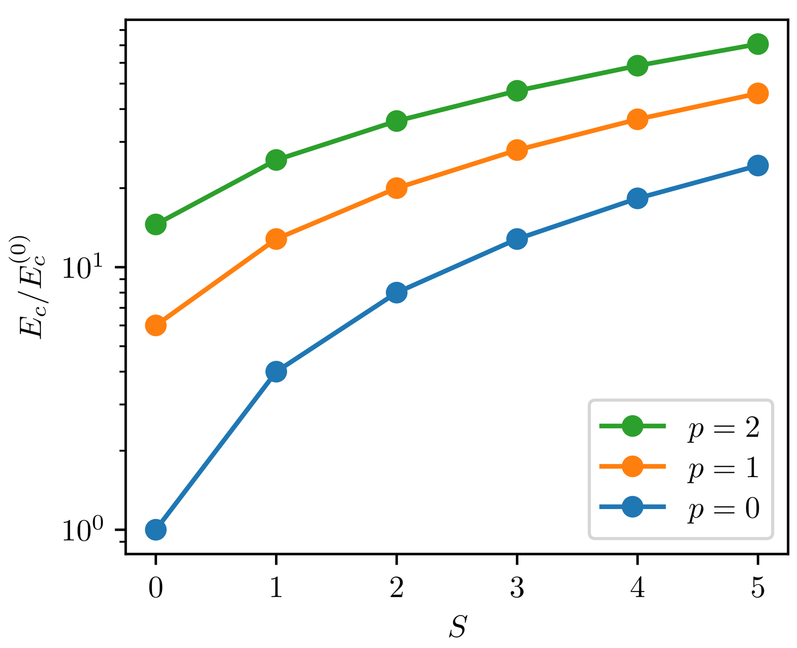


Figure 3. Critical pulse energy as a function of the mode numbers.

By integrating the stationary version NPSE, we obtain

$$\frac{da}{dt} = \sqrt{-V(a)}, \qquad V(a) = -\kappa_{pS}a^2 - a^2\sqrt{1-\gamma_{pS}a^2}.$$

Expanding to sixtic order, we obtain the potential of an equivalent **cubic-quintic** equation.

$$V_{3-5}(a) = -a^2\left[(\kappa_{pS}+1) - \frac{1}{2}\gamma_{pS}a^2 - \frac{1}{8}\gamma_{pS}^2a^4\right] .$$

Repeating the analysis for the energies, we obtain the cubic and cubic-quintic cases

$$E_3 = \frac{2\sqrt{2}}{\gamma_{pS}}\sqrt{1+\kappa_{pS}}, \qquad E_{3-5} = \frac{2}{\gamma_{pS}}\arctan(\sqrt{2}\sqrt{1+\kappa_{pS}}) .$$

We can apply the VK criterion to each of the cases.

	C	CC	NPSE
$\partial E/\partial\kappa_{pS}$	>0	>0	2 branches
E_C	$+\infty$	π/γ_{pS}	$4/(3\gamma_{pS})$

Table 1. Stability with the Vakhitov-Kolokolov criterion and critical energy for different quasi-one dimensional effective equations.

References

1. **F.L.** and L. Salasnich, Nanophotonics **14**, 6 805-813 (2025).
2. P. Parra-Rivas, Y. Sun, and S. Wabnitz, Opt. Commun. **546**, 129749 (2023).
3. L. Khaykovich, and B. A. Malomed, Phys. Rev. A **74**, 023607 (2006)
4. L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A **65**, 043614 (2002)
5. L. Salasnich, B. A. Malomed, and F. Toigo, Phys. Rev. A **76**, 6, 063614 (2007).
6. A. L. Gaeta, Phys. Rev. Lett. **84**, 3582 (2000).