Quantum Optics and Lasers - Homework 1

Francesco Lorenzi

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1 Fluctuation of the coherent state

Question Following the definition of the quadrature operators $\hat{X}_1$ and $\hat{X}_2$ and of their fluctuations, demonstrate that when the light in a single mode is in a coherent state $|\alpha\rangle$ then such fluctuations are the same as for the vacuum state (Eq. 3.16).

Answer We start recalling a fundamental properity of the coherent states: every coherent state is a right eigenstate of the annihilation operator $\hat{a}$ (this is the actual definition of a coherent state). So the most useful representation of the quadrature operators are the one which use the concepts of *creation* and *annihilation* operators,

$$\begin{aligned} \hat{X}_1 &= \frac{1}{2}(\hat{a} + \hat{a}^\dagger) \\ \hat{X}_2 &= \frac{1}{2i}(\hat{a} - \hat{a}^\dagger) \end{aligned} \tag{1}$$

As the fluctuations of these operators in the vacuum state are 1/4, we need to prove that this value is also valid for a generic coherent state $|\alpha\rangle$, so

$$\left\langle (\Delta \hat{X}_1)^2 \right\rangle_\alpha = \left\langle (\Delta \hat{X}_2)^2 \right\rangle_\alpha = 1/4$$

proof: We start by proving the above relation for $\hat{X}_1$. By expanding the definition of fluctuation we can deduce a useful result, analogous to the usual theorem for the variance

$$\left\langle (\Delta \hat{X}_1)^2 \right\rangle_\alpha = \left\langle \hat{X}_1^2 \right\rangle_\alpha - \left\langle \hat{X}_1 \right\rangle_\alpha^2$$

So expanding the bra-ket formalism for the expected value,

$$\left\langle (\Delta \hat{X}_1)^2 \right\rangle_\alpha = \langle\alpha| \hat{X}_1^2 |\alpha\rangle - \langle\alpha| \hat{X}_1 |\alpha\rangle^2 \tag{2}$$

Before analyzing the terms, lets recall two fundamental facts regarding the creation and annihilation operators:

1) $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \iff \langle\alpha| \hat{a}^\dagger = \langle\alpha| \alpha^*$ by the definition of coherent state

2) $\left[\hat{a}, \hat{a}^\dagger\right] = 1 \implies \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$ using the definition of commutator so, expanding further the right side of (2), using the definition (1) of the quadrature operators, we have for the first addend:

$$\begin{aligned}\langle\alpha|\,\hat{X}_1^2\,|\alpha\rangle &= \frac{1}{4}\,\langle\alpha|\,\hat{a}^2+\hat{a}^{\dagger 2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\,|\alpha\rangle\\ &= \frac{1}{4}\,\langle\alpha|\,\hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}^\dagger\hat{a}+1\,|\alpha\rangle\\ &= \frac{1}{4}\,\langle\alpha|\,\alpha^2+\alpha^{*2}+2\alpha^*\alpha+1\,|\alpha\rangle\\ &= \frac{1}{4}((\alpha+\alpha^*)^2+1)\\ &= \mathrm{Re}[\alpha]^2+\frac{1}{4}\end{aligned}$$

and for the second addend:

$$\begin{aligned}\langle\alpha|\,\hat{X}_1\,|\alpha\rangle^2 &= \frac{1}{4}\,\langle\alpha|\,\hat{a}+\hat{a}^\dagger\,|\alpha\rangle^2\\ &= \frac{1}{4}\,\langle\alpha|\,\alpha+\alpha^*\,|\alpha\rangle^2\\ &= \frac{1}{4}(\alpha+\alpha^*)^2\\ &= \mathrm{Re}[\alpha]^2\end{aligned}$$

The difference of the two terms gives the expected result:

$$\left\langle(\Delta\hat{X}_1)^2\right\rangle_\alpha = \frac{1}{4}.$$

Proceeding in evaluating the relation for $\hat{X}_2$, we find, after the same passages, that the first addend can be written as

$$\langle\alpha|\,\hat{X}_2^2\,|\alpha\rangle = \mathrm{Im}[\alpha]^2+\frac{1}{4}$$

whereas the second addend is

$$\langle\alpha|\,\hat{X}_2\,|\alpha\rangle^2 = \mathrm{Im}[\alpha]^2$$

so by subtracting them, as we have done for $\hat{X}_1$, the relationship in (Eq. 3.16) is finally proven.

2 Photon statistics

Question Consider a beam of blue light (405 nm).
a) What is the power that carry one billion photons per second?
b) What is the fluctuation of the arrival number for time bins of 10 ps duration? (considering unitary detection efficiency)
c)How much momentum is carried by one photon?

Answer a) The energy per photon is $E = h\nu = h\frac{c}{\lambda} = 3.061$ eV, so a flux of 10^9 photons per second carry the total power of

$$P = 10^9\cdot E = 3.063\cdot 10^9\mathrm{eV/s} = 490.5\mathrm{pW}.$$

b) Considering the coherent nature of this source (which is supposed to be in a coherent state), we have a Poissonian ($\sim P$) description of the arrival process. The number of arrivals in a time span of 10ps is described by a random variable $X\sim P(\lambda)$ where $\lambda = 10^9\mathrm{Hz}\cdot 10\mathrm{ps} = 0.01$ is the mean number of photons arriving in such a time span.

The expected fluctuation defined to be the variance, from random variable theory we have that $\text{var}(X) = \lambda = 0.01$ (we can even consider the standard deviation, as it has a more natural unit, which in this case is 0.1 *photons*)

c) The momentum is given by $|\mathbf{p}| = \hbar|\mathbf{k}| = h\frac{1}{\lambda} = 1.6361 \cdot 10^{-27} kg\, m/s$.

3 Temporal evolution of a single mode state

Question Consider a single-mode cavity field. At time t=0 it is expressed as the superposition of two Fock states, as:

$$|\psi(0)\rangle = (|n\rangle + e^{i\phi}|n+1\rangle)/\sqrt{2}$$

where the phase is $\phi \neq 0$ and $n > 0$. Describe the time evolution of this state, that is $|\psi(t)\rangle$ at a later time $t > 0$.

Answer Using the Schrödinger picture, we impose the condition on the evolved state as specified with the following equation

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle \tag{3}$$

We have a description of the state as a superposition of Fock states, so it is useful to express the Hamiltonian using the operator $\hat{a}^\dagger\hat{a}$: in fact, every Fock state is an eigenstate of the operator $\hat{a}^\dagger\hat{a}$, and the corresponding eigenvalue is the number of quanta of the state. For that reason the operator is called *number operator*, $\hat{n}$, and it is important to notice that it is an observable, because $(\hat{a}^\dagger\hat{a})^\dagger = \hat{a}^\dagger\hat{a}$. The Hamiltonian can indeed be expressed in the following, highly intuitive, way

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

The solution of the equation (3) is carried out with respect to the initial state using this relation

$$|\psi(t)\rangle = \exp\left[-\frac{i\hat{H}t}{\hbar}\right]|\psi(0)\rangle \tag{4}$$

(more comments on the solution can be found in the appendix), so expressing the right term of the equation (4) with the definition of the state,

$$|\psi(t)\rangle = \exp\left[-i\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)t\right]\frac{|n\rangle + e^{i\phi}|n+1\rangle}{\sqrt{2}}$$

the above expression can successfully be written in an explicit form, by means of the eigenstate-eigenvector properties of the number operator. In fact, it can be proven with little effort that an analytic function of a given observable has the same eigenstates of the observable itself, and the corresponding eigenvalues are the original eigenvalues transformed by that function (motivation in the appendix). By using this fact, the following time evolution is finally constructed as follows

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(\exp\left[-i\omega\left(n+\frac{1}{2}\right)t\right]|n\rangle + \exp\left[-i\left(\omega\left(n+\frac{3}{2}\right)t - \phi\right)\right]|n+1\rangle\right).$$

An important point is to observe that this solution indicates that the state $|\psi(0)\rangle$ is actually not a stationary state, because of the fact that it is a superposition of eigenstates of an operator ($\hat{H}$) which are relative to different eigenvalues, and so is not an eigenstate for the same operator.

4 Photon absorption in Fock basis

Question Consider the superposition of the vacuum and 20 photon number state

$$|\psi\rangle = (|0\rangle + |20\rangle)/\sqrt{2}$$

1. Calculate the average photon number for this state.
2. Now, the annihilation operator is applied to the state. So a single photon is absorbed. Calculate the new value for the average photon number.
3. The annihilation operator is applies again to the resulting state. Calculate the new value for the average photon number.
4. Analyse two results and the meaning of the average photon number.

Answer 1. The calculation of the average number of quanta is done using the fact that, as discussed in question 3), a Fock state is an eigenstate of the number operator, for which the corresponding eigenvalue is the *well determined number* of quanta of such state. This concept can be applied for mixed states, for which the number of quanta is actually *not* well determined, and so the procedure gives the *average* photon number. In the case of the (already normalized) state $|\psi\rangle$ this is evaluated as

$$\begin{aligned}
\langle \hat{n} \rangle_\psi = \left\langle \hat{a}^\dagger \hat{a} \right\rangle_\psi &= \langle \psi | \, \hat{a}^\dagger \hat{a} \, | \psi \rangle \\
&= \frac{1}{2}(\langle 0 | + \langle 20 |) \, \hat{a}^\dagger \hat{a} \, (|0\rangle + |20\rangle) \\
&= \frac{1}{2}(\langle 0 | \; \hat{a}^\dagger \hat{a} \; |0\rangle + \langle 20 | \; \hat{a}^\dagger \hat{a} \; |20\rangle) \qquad \text{using the orthogonality of Fock states} \\
&= \frac{1}{2}(0 + 20) = 10
\end{aligned}$$

2. After the application of the annihilation operator, the state becomes:

$$|\psi'\rangle = \hat{a} \, |\psi\rangle = \hat{a} \frac{|0\rangle + |20\rangle}{\sqrt{2}} = \frac{\sqrt{20} \, |19\rangle}{\sqrt{2}}$$

The corresponding average number of quanta is calculated having care of the normalization of $|\psi'\rangle$, the calculations are even simpler than before because the new state is a pure Fock state

$$\langle \hat{n} \rangle_{\psi'} = \frac{\langle \psi' | \, \hat{n} \, | \psi' \rangle}{\langle \psi' | \psi' \rangle} = 19$$

3. After a subsequent application of the annihilation operator (we consider it done on the normalized version of $|\psi'\rangle$ for simplicity), the state becomes

$$|\psi''\rangle = \hat{a} \left| \psi' \right\rangle = \hat{a} \, |19\rangle = \sqrt{19} \, |18\rangle$$

which gives an average number of quanta of

$$\langle \hat{n} \rangle_{\psi''} = 18.$$

4. At first sight, the properties of the annihilation operator on Fock states, not to mention its actual name, suggest that its action on the state is to "decrease the number of quanta by one unit". But the superposition postulate undermines this naïf concept: the number of quanta in a mixed state is *not well defined*, and the only thing that can be said about such number is the expected value.

The annihilation operator doesn't always decrease the average number of quanta, as for the case of the first application, in which it increases it. This is because of the very uncertainty of the first state which is composed by a superposition.

A more effective argument can be found by trying to express the average number of photons of a generic state $|\omega'\rangle = \hat{a}\,|\omega\rangle$, *after* an application of the annihilation operator. Working out the algebra, remembering to normalize by the norm of $|\omega'\rangle$ when computing expected values, we have the following way to express the average number

$$\begin{aligned}
\langle\hat{n}\rangle_{\omega'} &= \frac{\langle\omega'|\,\hat{n}\,|\omega'\rangle}{\langle\omega'|\omega'\rangle} \\
&= \frac{\langle\omega|\,\hat{a}^\dagger\hat{n}\hat{a}\,|\omega\rangle}{\langle\omega|\,\hat{a}^\dagger\hat{a}\,|\omega\rangle} \\
&= \frac{\langle\omega|\,\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\,|\omega\rangle}{\langle\omega|\,\hat{n}\,|\omega\rangle} && \text{using } \hat{n} = \hat{a}^\dagger\hat{a} \\
&= \frac{\langle\omega|\,\hat{a}^\dagger(\hat{a}\hat{a}^\dagger - 1)\hat{a}\,|\omega\rangle}{\langle\omega|\,\hat{n}\,|\omega\rangle} && \text{using } \hat{a}^\dagger\hat{a} = \hat{a}\hat{a}^\dagger - 1 \\
&= \frac{\langle\omega|\,\hat{n}^2 - \hat{n}\,|\omega\rangle}{\langle\omega|\,\hat{n}\,|\omega\rangle} \\
&= \frac{\langle\hat{n}^2\rangle_\omega - \langle\hat{n}\rangle_\omega}{\langle\hat{n}\rangle_\omega} && \text{using linearity}
\end{aligned}$$

At this point we can define a quantity similar to the fluctuation used in answer 1), relative to the number of a state $|\omega\rangle$

$$\left\langle(\Delta\hat{n})^2\right\rangle_\omega := \left\langle(\hat{n} - \langle\hat{n}\rangle_\omega)^2\right\rangle_\omega = \left\langle\hat{n}^2\right\rangle_\omega - \langle\hat{n}\rangle_\omega^2$$

By reformulating slightly the expression of $\langle\hat{n}\rangle_{\omega'}$, we deduce

$$\langle\hat{n}\rangle_{\omega'} = \frac{\langle\hat{n}^2\rangle_\omega - \langle\hat{n}\rangle_\omega}{\langle\hat{n}\rangle_\omega} = \frac{\langle\hat{n}^2\rangle_\omega}{\langle\hat{n}\rangle_\omega} - 1 = \frac{\langle\hat{n}^2\rangle_\omega - \langle\hat{n}\rangle_\omega^2}{\langle\hat{n}\rangle_\omega} + \langle\hat{n}\rangle_\omega - 1$$

finally, using the above definition of the fluctuation of the number:

$$\langle\hat{n}\rangle_{\omega'} = \underbrace{\langle\hat{n}\rangle_\omega - 1}_{\text{naïf decreasing}} + \underbrace{\frac{\left\langle(\Delta\hat{n})^2\right\rangle_\omega}{\langle\hat{n}\rangle_\omega}}_{\text{effect of the indeterminacy}}$$

the physical meaning of the above expression is very interesting: the expected number *after* the application of $\hat{a}$ differs from $(\langle\hat{n}\rangle_\omega - 1)$, which is the naïf decreasing of the number, by a term proportional to the just defined fluctuations of the number in the original state.

In the case of analysis:

$$\begin{aligned}
\left\langle(\Delta\hat{n})^2\right\rangle_\psi &= \frac{20^2}{2} - 10^2 = 100 && \text{mixed state} \\
\left\langle(\Delta\hat{n})^2\right\rangle_{\psi'} &= \left\langle(\Delta\hat{n})^2\right\rangle_{\psi''} = 0 && \text{pure states}
\end{aligned}$$

so the above argument give some insight in the seemingly bizarre sequence [10, 19, 18] of expected numbers.

Appendix: mathematical comments on answer 3)

Handling of operator algebra The solution of the Schrödinger equation (3) can be deduced using the very deep language of operator theory, in particular referring to the so called *shift operator* theorem, which states that, under some regularity assumptions:

If $\dfrac{df}{dt}(t) = \hat{O}f(t)$, where $\hat{O}$ is a generic operator defined over the space of all functions, then $f(t+h) = e^{\hat{O}h}f(t) \qquad h \in \mathbb{R}$

the proof of this theorem is done handling the operators in symbolic form by means of the Taylor expansion, and in particular, relies on the fact that, formally:

$$f(t+h) = e^{\frac{d}{dt}h}f(t)$$

By analyzing the equation (3) using this point of view, we deduce the so called propagator $U(t)$

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle \qquad \text{where } U(t) = \exp\left[-\frac{i\hat{H}t}{\hbar}\right]$$

For a comprehensive discussion, refer to *K. Yosida*, Operational calculus, pg.74.

Functions of observables If the function of the observable $f(\hat{O})$ is analytic, with power series expansion

$$f(\hat{O}) = \sum_{k=0}^{\infty} c_k \hat{O}^k$$

and if $|\omega\rangle$ is an eigenstate of the operator $\hat{O}$, with eigenvalue ω, using the power series expansion of f we get:

$$f(\hat{O})\,|\omega\rangle = \sum_{k=0}^{\infty} c_k \hat{O}^k\,|\omega\rangle = \sum_{k=0}^{\infty} c_k \omega^k\,|\omega\rangle = f(\omega)\,|\omega\rangle$$

so the eigenstates of the observable $f(\hat{O})$ are the same of the operator $\hat{O}$, and the corresponding eigenvalues are the images of the original eigenvalues by f.

For a more general discussion refer to *P.A.M. Dirac*, The principles of Quantum Mechanics, §11.11